COMPARISONS

OF

STANDARDS OF LENGTH

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OF THE

STANDARDS OF LENGTH

OF

ENGLAND, FRANCE, BELGIUM, PRUSSIA, RUSSIA, INDIA, AUSTRALIA,

MADE AT THE

ORDNANCE SURVEY OFFICE, SOUTHAMPTON,

BY

CAPTAIN A. R. CLARKE, R.E., F.R.S.,

UNDER THE DIRECTION OF

COLONEL SIR HENRY JAMES, R.E., F.R.S, &c.

COR: MEM: OF THE ROYAL GEOGRAPHICAL SOCIETY OF BERLIN,
DIRECTOR OF THE ORDNANCE SURVEY.

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ERRATA.

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24	12	$\frac{2\,b\,b'}{b + b'} \quad \frac{a}{4}$	$\frac{2\,b\,b'}{b + b'} - \frac{a}{4}$
24	12	$\frac{b' - b}{b' + b}\,\frac{x^2}{2} \quad \frac{x^3}{3\,a}$	$\frac{b' - b}{b' + b}\,\frac{x^2}{2} - \frac{x^3}{3\,a}$
24	17	$\frac{i}{u}$	$\frac{i}{\mu}$
24	18	$\frac{\delta}{u} + \frac{i}{u}\,x$	$\frac{\delta}{\mu} + \frac{i}{\mu}\,x$
25	last	$\frac{dy^2}{x^2}\,dx$	$\frac{dy^2}{dx^2}\,dx$
64	16	first	just.
138	16	$(2 \cdot 1315208 \pm \ldots .)$	$(2 \cdot 13152080 \pm \ldots .)\ Y_{55}$
143	last	$\frac{4 \cdot 74}{3600}$	$\frac{474}{3600}$
145	28	partical	particle.
162	20	$2\,Y_{67}$	Y_{67}
192	26	to be nearly	to be as nearly.

PREFACE.

The Figure and Dimensions of the Earth have been determined in this country by the labors of G. B. Airy, Esq., Astronomer Royal of England,* but, on the completion of the Principal Triangulation of the United Kingdom, we gave the Figure and Dimensions as derived from our own geodetic operations, and also the result derived from the combination of all the separate measurements of Arcs of Meridians in Peru, France, Prussia, Russia, Cape of Good Hope, India, and in the United Kingdom.† From which we derived the following dimensions :—

Equatorial semi-diameter	20926330
Polar semi-diameter	20855240
Ellipticity	$\frac{1}{294 \cdot 36}$

We also published in 1858 Geodetical Tables based on the above elements of the Figure of the Earth, but these numbers being dependent upon old comparisons, must now be slightly modified in conformity to the results of the comparisons now made.

In computing the Figures of the Meridians and of the Equator for the several measured Arcs of Meridian, it is found that the equator is slightly elliptical, having the longer diameter of the ellipse in $15° \cdot 34'$ east longitude and the shorter in $105° \cdot 34'$ east longitude. In the Eastern hemisphere the meridian of $15° \cdot 34'$ passes through Spitzbergen, a little to the west of Vienna, through the Straits of Messina, through Lake Chad in North Africa, and along the west coast of South Africa, nearly corresponding to the meridian which passes over the greatest quantity of land in that hemisphere. In the western hemisphere this meridian passes through Bering's Straits, and through the centre of the Pacific Ocean, nearly corresponding to the meridian which passes over the greatest quantity of water in that hemisphere.

The meridian of $105° \cdot 34'$ passes near North East Cape in the Arctic Sea, through Tonquin and the Straits of Sunda, and corresponds nearly to the meridian which passes over the greatest quantity of land in Asia, and in the western hemisphere it passes through

* Encyclopædia Metropolitana. Art. Figure of the Earth.

† Account of the Principal Triangulation, pp. 733—778.

Smith's Sound in Bering's Straits, near Montreal, near New York, between Cuba and St. Domingo and close along the western coast of South America, corresponding nearly to the meridian passing over the greatest amount of land in the western hemisphere.

These meridians therefore correspond with the most remarkable physical features of the globe.

	Feet.
The longest semi-diameter of the equatorial ellipse is... ...	20926350
and the shortest	20919972
giving an ellipticity of the equator equal to	$\frac{1}{3269 \cdot 5}$
The polar semi-diameter is equal to	20853429
The maximum and minimum polar compressions are... ...	$\frac{1}{285 \cdot 97}$ and $\frac{1}{313 \cdot 38}$
Or a mean compression of very closely	$\frac{1}{300}$

The Specific Gravity of the Earth was determined by Dr. Maskelyne in 1774 from observations on the attraction of the mountain Schehalien in Scotland, and was found to be 4·90, and in 1855 I had observations made at Arthur's Seat near Edinburgh, and the Specific Gravity of the Earth obtained from the attraction of that mountain was found to be 5·316.*

The Principal Triangulation of the United Kingdom was finished in 1851, and the Triangulations of France, Belgium, Prussia, and Russia were so far advanced in 1860, that, if connected, we should have a continuous Triangulation from the Island of Valentia on the south-west extremity of Ireland, in north latitude 51° 55′ 20″ and longitude 10° 20′ 40″ west of Greenwich, to Orsk on the River Ural in Russia.

It was, therefore, possible to measure the length of an arc of parallel in latitude 52° of about 75°, and to determine, by the assistance of the electric telegraph, the exact difference of longitude between the extremities of this arc, and thus obtain a crucial test of the accuracy of the Figure and Dimensions of the Earth as derived from the Measurement of Arcs of Meridian, or the data for modifying the results previously arrived at.

The Russian Government, therefore, at the instance of M. Otto Struve, Imperial Astronomer of Russia, invited in 1860 the co-operation of the Governments of Prussia, Belgium, France, and England to effect this most important object, and to their great honor they all consented, and granted the necessary funds for the execution of the work.

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* Philosophical Transactions, 1856, page 591.

The portion of the work which was assigned to me was the connection of the Triangulation of England with that of France and Belgium, and I published the results of this operation in 1862.* But this work has been done in duplicate, for when application was made to the French Government to permit the necessary observations to be made in France, they not only consented to allow this, but at the same time volunteered to join in the labor and expense of the work itself.

It would obviously have been wrong to mix up observations made with different kinds of instruments, and on different principles, and, therefore, it was agreed that the work should in fact be made in duplicate, both the French and English geometricians using the exact same stations.

The results obtained by the French geometricians is published in the Supplement to Vol. IX. of the "Memorial du Dépôt Général de la Guerre," 1865, and the agreement with the results obtained by the English is truly surprising.

But however accurately the trigonometrical observations might be performed, it is obvious that without a knowledge of the exact relative lengths of the Standards used as the units of measure in the triangulation of the several countries, it would be impossible accurately to express the length of the arc of parallel in terms of any one of the Standards.

It was, therefore, necessary that a comparison of the Standards of Length should be made, and as we had a building and apparatus expressly erected for the purpose of comparing Standards at this office, the English Government, on my recommendation, invited the Governments of the several countries named to send their Standards here, and we have had the following compared with the greatest accuracy :—

1. Russian Standard double Toise **P**.
2. Prussian Standard Toise.
3. Belgian Standard Toise.
4. Platinum Metre of the Royal Society, compared with the Standard Metre of France, by M. Arago.
5. English Standard Yards, A, B, C, 29, 47, 51, 55, 58.
6. Ordnance Survey 10-feet Standard Bar.
7. Indian 10-feet Standard Bars, new and old.
8. Australian 10-feet Standard Bars.
9. In addition to the above, the 10-feet Standard Bar of the Cape of Good Hope was compared here in 1844.

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* "Extension of the Triangulation of the Ordnance Survey into France and Belgium." London, 1863.

We have invited the Governments of Austria, Spain, and the United States of America, also to send their Standards. We have been promised that of Austria, and but for the unfortunate war in which she has been engaged we should have received it before this.

I have entrusted the execution of the work of comparison, and the drawing up of the results, to Captain Alexander R. Clarke of the Royal Engineers, who also designed the greater part of the apparatus used. The numerous comparisons to be made entailed a great amount of labor upon him, and his assistants, Quartermaster Steel and Corporal Compton of the Royal Engineers.

Before the connection of the Triangulation of the several countries into one great net-work of triangles extending across the entire breadth of Europe, and before the discovery of the Electric Telegraph, and its extension from Valentia to the Ural Mountains, it was not possible to execute so vast an undertaking as that which is now in progress. It is in fact a work which could not possibly have been executed at any earlier period in the history of the world. The exact determination of the Figure and Dimensions of the Earth has been the great aim of Astronomers for upwards of two thousand years, and it is fortunate that we live in a time when men are so enlightened as to combine their labors to effect an object desired by all, and at the first moment when it was possible to execute it.

HENRY JAMES,
Colonel Royal Engineers.

Ordnance Survey Office,
Southampton, 10th August 1866.

I.

DESCRIPTION OF THE COMPARING APPARATUS

AND OF CERTAIN

STANDARD BARS.

1.

The room especially designed and built for the purpose of comparing standard measures is situated in one of the angles of the boundary wall of the grounds of the Ordnance Survey Office, Southampton. The situation has two advantages; it is sheltered in some measure from the mid-day sun by a high house immediately on the south side, and the direction of the room is very nearly perpendicular to the meridian. The bar-room proper is built inside of an outer building, and partly sunk below the level of the ground. This will be understood by a reference to Plates I. and II.

FFF . . . . . is the old boundary wall.
ffff . . . . . the walls of the outer building.
HHHH . . . the walls of the inner, or bar-room.
ppp . . . . . . a clear passage round the inner room.
rr . . . . the concrete roof of the room.
www . . . . . the windows of the outer room.
WW . . . . . the two windows of the inner room.
ss . . . . steps down to the floor of inner room.
DD′D, ... the double door of inner room.
uu′ . . . . . . strong wooden shutters of windows W.
PP′P″ . . . . stone piers for carrying microscopes.
BB . . . . . . a hollow beam of mahogany for carrying the bars.
cc . . . . two strong fir rafters resting on
SS . . . . two stones, deeply sunk, carrying one end of each rafter.
q . . . . a small block of stone supporting the other end of each rafter.
dd . . . . two oak posts fixing the rafters, and transferring any upward pressure to the ceiling.
ee . . . . two blocks of wood firmly screwed to the rafters *cc*, and which immediately receive the hollow beam B.
g . . . . a gas burner; one opposite each window W.

On the two sides on which the wall *ff* is in contact with the boundary wall, it is in a series of arches as shown on the left of Plate II. The walls of the inner room will be seen from the plates to be *double;* the outer is two feet in thickness; then a space of three inches of air; then an inside wall of four inches or a single brick. The passage *p* is of sufficient breadth to permit one to pass round. The roof of the bar-room is supported by three iron girders, which carry on their lower flanges large slabs of slate covering the whole room; on these again is laid concrete, one foot in thickness. There are three orifices in the roof of this room; two conical, over the two gas burners, and one cylindrical, for the passage of a stove pipe through the roof. The conical openings are in the middle of the breadth of the room, and about four feet from each end of the room; the upper and outer diameter of the orifice is three inches, the lower and inner about 12 inches. The cylindrical orifice is in the centre of the length of the room, but near the side of the room furthest from the piers PP′P″. No stove is ever now used; but after the room was first built a stove was kept burning for some months for the purpose of drying. The gas burners are sometimes used when there is no work going on, to promote ventilation, and carry away any dampness. The room, however, is generally free from damp, and it is very desirable to keep it so.

The foundations of the walls H are strongly built. The flooring consists, first, of low brickwork arches, running the length of the room; then on these is laid concrete, forming a plane floor, interrupted by a space around the microscope piers PP′P″, and the stones S, S. On the concrete floor, again, is laid a board flooring in three pieces, as seen in Plate I.; each of these is framed with rafters underneath (see Plate III. *jjj*.) These three pieces of flooring are disconnected with the walls of the room, and each rests on four blocks of india-rubber which lie between the woodwork and the concrete. By this means a steady bearing was obtained, and any jar or shake communicated to the boards of the floor is deadened in its transmission to the concrete. Thus the vibration caused by any person moving about on the floor is not sensibly transmitted to the piers PP′P″, which are entirely disconnected from the concrete floor, and have a foundation of their own.

The fir rafter *cc*, which measures nine inches square in section, is bolted down to the stone S, which is sunk four feet deep in the earth. The rafter has no contact with the concrete floor; its second point of support is at its further extremity where it rests on a small block *q*, and is kept down in its place rigidly by a post of oak, which is jammed against the roof. Neither the board flooring nor the walls have any contact with *cc*, and the stone S has no contact either with the piers P or the concrete floor. The hollow mahogany beam BB is entirely supported by the overhanging part of *cc*. It will be seen by this, that changes in the weight supported on BB will not affect the piers PP′P″; so that if we remove a light bar, and replace it by a heavy one, the microscopes will not be affected.

The windows WW are opposite to two of the windows *ww* of the outer building. The room is but moderately lighted by daylight, the points of the bars being always illuminated by the flame of a candle condensed. The windows are closed by thick wooden doors on the outside. The floor of the bar-room being two feet beneath the surface of the external ground is reached by the steps *s*, *s*. The room is entered by the folding doors D′D′ on the outside, and the door D on the inside. A person entering the room has sufficient space to close the door D′D′ after him before he opens the door D. Thus very little of the external air is brought in.

The object of the different arrangements so far, it will be seen, is to provide a room which shall not be subject to any sudden changes of temperature, and which shall afford three entirely independent foundations for, (1) the observer, (2) the bars compared, (3) the micrometer microscopes.

The dimensions (internal) of the bar-room are—length 19·7 feet, breadth 11·5, and height 7·7 feet.

Plate III. presents a front view of the apparatus, with a ten-feet bar under comparison. The flooring is shown in section, and the front side of the box containing the bar is supposed removed. Plate IV. is an end view of the same, and shows the ten-feet bar and the standard yard; the end of the box being supposed removed. Originally, the outer piers were square in plan as shown in Plate I., but were subsequently altered to the shape shown in Plate III. in order that a double toise might be compared.

pppp . . . . is the centre stone pier, 4 feet in length.
p′p′p′ . . . . is the left stone pier.
p″p″p″ . . . is the right stone pier.
CCC . . . . is the concrete flooring cut away from below *c,c,*.
jjj . . . . . the joists of the board flooring resting on india-rubber blocks.
BBB . . . . the hollow mahogany beam.
a,a, ... slips of mahogany screwed to the upper surface of BB.
a′a . . . . . rails of cast-iron, planed.
gg, *g′g′* . . carriages running on the rails.
ooo . . . . . the box containing the bars, having a double bottom *o′o′*.
bb′ . . . . . . the bar resting on the eight rollers *rrr*...
tttt . . . . . . thermometers whose bulbs are bent down into wells in the bar.
nnnn . . . . microscopes for reading the thermometers.
mm . . . . . micrometer microscopes.

We shall now describe more in detail the different parts.

The Mahogany Beam.

Great rigidity is here requisite, so that no flexure shall be perceptible under the weight of the bar. The four planks of mahogany of which this beam is built were very carefully selected from well-seasoned timber. The upper and lower planks are 14 inches wide by 2 inches thick; the two sides are 3 inches thick by 5 inches wide. These four, being very truly planed and fitted, were fastened together by copper screws. On the upper surface of the mahogany beam, and running its whole length, are planed cast-iron rails *a′a* (each in two pieces, seven feet long). As the beam BB may naturally be expected to be liable to warp in course of time, two strips of mahogany *a,a,* are screwed along the whole length of the beam, and their upper surfaces very carefully planed, to receive the planed iron rails. If the beam BB be found to warp, the upper surface of the strips *a,a,* can be planed true again. It will be observed that the inner and outer rails differ in section. The beam BB rests upon the blocks *ee* (which are strongly fastened to the rafters *cc*). It simply rests by its own weight, which is considerable, and is not fastened. Great care is taken that it is (1) horizontal, and (2) has a perfectly steady bearing on its supports, which are seven feet asunder.

The Carriages.

The two carriages marked *gg*, *g′g′*, which support the box that holds the standard bars in Plate III., will be seen in more detail in Plates IV. and V. Fig. 3, Plate V., is an isometric projection, which serves to give a general idea of the carriage as it stands on the rails. It consists essentially of four parts, (1) a lower carriage *k′k,* which runs on three wheels, along the rails *a′a*; (2) an upper carriage *l′l,* which runs on three wheels on rails planed out of the upper surface of the lower carriage perpendicular to the direction of the iron rails *a′a*; (3) a clamp *ss*, with slow motion screw σ and a T-shaped piece *ttt′*.

The lower carriage is a brass casting, strongly ribbed on the under side, and having bearings for the three wheels m_1 m_2 m_3; on the upper side are left three rails, r_1 r_2 r_3, for the wheels of the upper carriage, and two grooved rails s_1 s_2 for the clamp. These rails are very carefully planed. The wheels m_1 m_2 m_3 have axles fitted to the bearings. The wheels m_1 m_3 are grooved to run on the angular rail *a′*, and m_2 is plain, running on the flat rail *a*. By this arrangement it will be seen that the carriage cannot take a faulty bearing. The wheel m_2 is provided with a brake *xx,y*, which acts as a clamp, and prevents the carriage running on the rails *a′a*.

The motion of the upper carriage is perpendicular to that of the lower. The body of this carriage is a casting somewhat similar to the lower, strongly ribbed below, and provided with bearings for the three wheels n_1 n_2 n_3. These three wheels are all grooved, running upon the angular rails planed on the upper surface of the lower carriage. In order to ensure a perfect bearing upon the rails at all times, one of the wheels, n_1, admits of a very small play in a transverse direction, that is, its axle can slide parallel to its own length *in* its bearings. By this means a perfect bearing is secured on the supposition of there being any defect of parallelism in the rails $r_1 r_2 r_3$. Screwed to the upper surface of the upper carriage are two bearings t_1 t_2, which receive the turned pivots or arms of the piece ttt'. The boss u receives the box which carries the bars. The extremity t' of the T-shaped piece rests upon the top of a screw q, which is screwed up from below through the body of the upper carriage. By means of this screw the piece ttt' is made truly level in a direction perpendicular to the rails $a'a$. It will be seen that there can be no *shake* in the piece ttt', but it must present a perfectly steady basis for the bar box to rest upon, while motion is communicated to it in two directions, and these with perfect smoothness.

It is necessary to be able to communicate a slow motion to this upper carriage, and for this purpose a clamp s is provided, running upon the grooved clamping rails s_1 s_2. Upon these grooved rails slips of brass are screwed, upon which the clamp bites. The mode of clamping will be understood best perhaps from Figure 4, Plate V. The screw s' serves to draw together the piece s and a lower plate c which slides in the grooves. Thus s becomes a fixture at any required part of the rails, and the slow motion screw σ, working in a female screw ν attached to the upper carriage imparts to it a fine motion. The screw σ has a ball joint at b, and is kept in its socket by a small plate ee above. This small plate may be screwed down with more or less pressure, as may be required. The play of the upper carriage on its rails is about three inches; it is prevented running off behind by stops v_1 v_2.

To support the box containing the bar two of these carriages are required. They are precisely similar, with the exception that while the cross piece ttt' in one has a single boss in its centre, that of the other has two bosses, one at each extremity. Thus the box is supported on three points; by the one carriage it is supported on two points near the outer edges of the box, and by the other carriage it is supported in one point at the centre of its breadth. The carriage seen in Plate IV., and in Figures 1, 3, Plate V., is that one which has the single boss; but besides this single boss u two smaller ones, u_1 u_2, will be seen. These, however, are not rigid; they are pressed upwards by a *spring*, and merely serve to relieve u of part of the weight of the box.

Bar Boxes.

In Plate IV. we have an end view or section of the box containing the ten-feet bar **O1**. Its length is 10·6 feet. The mahogany of which it is made is one inch in thickness. The internal breadth is six inches, and depth six and a quarter. In order to rigidity, which is very necessary, there is a double bottom o o', a space of half an inch being left between the planks. The standard yard being of smaller dimensions than the bar **O1**, a strip of mahogany f is fastened to the inside of the box along its whole length. To the bottom o' of the box are strongly screwed down (Plates III., IV.) brass pieces dd, which carry the axles or pivots of the cradles which support the bar. Inside of each of the upright pieces dd, as seen in the end view, Plate IV., is a plate of brass fitted to slide vertically up and down; in these pieces are the Y's or bearings which take the pivots of the axles of the cradles. The sliding pieces are moved up and down by the thread of a vertical screw, which has a fixed shoulder at the top of the upright d. The upper parts kk of these screws are filed *square* (that is in section), so that they may be turned by a key similar to a watch key. It is of course necessary that the two extremities of the axle of the cradle be raised together, or that the screws k be always turned exactly the same amount and simultaneously. To effect this a key shown in Figure 6, Plate VI., is used. Here, by a system of three toothed wheels, when the milled head m is turned, it will be seen that the keys k, k' (which fit on to the screw heads kk, Plates III., IV.) are necessarily turned *together*.

The cover of the box is provided with apertures for permitting (1) the reading of the dots on the bar; (2) the thermometers; and (3) the use of the levelling key, so that the bar can be adjusted to focus without opening the box.

The Cradles.

In Plate III. it will be seen that the bar *bb′* rests upon eight rollers, which are framed in two systems of levers. $\beta\beta'$ is one lever, moveable round an axis through its centre, which axis is held in the support *d*. Inside (Plate IV.) of this lever are two smaller levers $\delta\delta$, each carrying at its extremity a roller *r*; the pivots of these inner and smaller levers are seen at *i* and *i*. In the construction care is taken that all the levers balance accurately on their pivots or axles, and that the rollers revolve with the least possible amount of friction.

It will be seen from this lever system that the pressure upwards of each roller upon the bar is the same. Thus the bar is supported by eight equal pressures, and applied at equal intervals. The proper interval between rollers of such a system is investigated by Mr. Airy, in a paper which will be found in the fifteenth volume of the Memoirs of the Royal Astronomical Society.

Each of the systems of levers $\beta\beta'$, $\gamma\gamma'$, can, as we have seen, be independently raised or lowered by the key (Figure 6, Plate VI.) Thus the bar *bb′* may be levelled, or either extremity adjusted to focus under the microscopes *mm*.

Of the eight rollers supporting the bar one only is a true cylinder; the remainder are slightly convex or barrel shaped. By this means a proper bearing is ensured. If the rollers were all true cylinders there would be a great probability of some of the contacts being at the outer edges of the bar. With the convex form the contact cannot be far from the centre of the breadth of the bar.

Support of the Standard Yard.

This is shown in end view, *f*, in Plate IV. and in side view, *bb′*, in Plate VI., Figure 5. It is a heavy cast-iron frame, upon which are screwed the brass supports $d_{\prime}d_{\prime}$ (IV.) or *cc′* (VI.) These carry two rollers $rr_{\prime}$ (IV.), which can be elevated or depressed by means of a key similar to that shown in Figure 6, Plate VI. The rollers, of which one is a cylinder, and the other slightly convex or barrel shaped, are 18 inches apart.

In the standard yard the lines marking the measure at either extremity are drawn on the surface of a gold pin at the bottom of a circular hole or well, drilled half way through the bar. In order to illuminate the lines properly for observation it is necessary to incline the bar slightly, as shown in the end view $b_{\prime}$ (Plate IV.) It is kept in this position by two small pieces of brass, one of which will be seen in the drawing, placed between the bar and the rollers. The measure of the inclination is the fraction $\frac{1}{8}$ or the angle $\tan^{-1}\frac{1}{8}$.

Micrometer Microscopes.

The two principal ones, marked H and K, and seen in the front view in Plate III., are shown in more detail in Plates IV. and VI. Figure 3, Plate VI., shows one of these with its turned collars $c'c_{\prime}$ and attached level *l*. The length of the tube from the diaphragm to the object glass is 12 inches, from the object glass to the external focus 3 inches. The magnifying power is about 60, and the value of a division of the micrometer about the thirty-five thousandth part of an inch.

The microscope is held by its collars in a gun-metal holder $\epsilon\epsilon$, Plate VI., Figures 1, 2, 4, and Plate IV. This holder is a hollow cylinder, having three arms $\epsilon_1\epsilon_2\epsilon_3$ at the middle of its length, and at either extremity internal bearings $e_1e_2e_3$, Figure 4, for receiving the microscope collars. The microscope is pressed, above and below, into its bearings, by a piece η, which is urged by a spring ζ, which can be drawn back at pleasure by the small milled head at ζ. The microscope is free to revolve in its bearings, but there is no shake. The bearings, as will be seen in Figure 4, are segments of circles.

This gun-metal holder is again supported on a cast-iron plate *aaa*, which rests, having three bosses on its under side, on the stone pier. Three screws $\beta\beta\beta$ are bolted into the cast-iron plate, and firmly fixed, so as to be truly vertical when the plate is placed on a horizontal plane. On each of these screws are two brass nuts $\gamma_{,}\gamma'$, and between them are held the arms of the gun-metal holder, which have holes through which the three screws pass. If the upper nuts $\gamma'\gamma'\gamma'$ be released, the weight of the microscope and its holder is supported by the three lower nuts $\gamma_{,}\gamma_{,}\gamma_{,}$, and by them the microscope may be levelled to any degree of nicety, and then by a gentle pressure of the upper nuts held down in its place. The iron casting *aaa* is provided with a hole $\delta\delta$ through which the gun-metal holder (with the lower spring ζ) passes.

It will be seen that by this arrangement a very important point is secured; namely, that the microscope is held in its holder without a strain, and the holder is itself held on the iron casting without a strain; while by means of the attached level, the microscope, revolving in its bearings, can be made vertical. Further, the means is supplied of raising or lowering, by the nuts, the microscope, when it may be required to bring it to focus over a given point.

The upper collar c' is furnished with a flange *f*, which takes a bearing on the upper edge of the gun-metal holder, and gives the microscope a definite position vertically.

Besides these microscopes there are smaller ones, of which one, A, is shown in Figures 5, 6, 7, Plate V. They are held in almost exactly the same manner as has just been described for the microscopes H and K.

Illumination.

The divided surface under observation is illuminated by the light of a candle CC, Plate IV., passing through a lens l' three inches in diameter; the lens is set in a brass cell, fitting into a corresponding circular orifice in the cast-iron stand. Thus the light of the candle is condensed on the point under observation (Fig. 5, Plate V.) The candle itself is contained in a brass tube which has a spring in its bottom continually pressing up the candle as it burns; thus the position of the flame is invariable, and when once lighted and placed in the proper position it does not require any further attention. The light of the candle illuminates also the micrometer head.

The light thus thrown on the divided surface of the bar is excellent, and leaves nothing to be desired. The heat of a candle is considerably less than the heat of gas or even of an oil lamp. Also the *position* of the flame with respect to the bar is favourable, for being above it, the air immediately heated by the candle ascends. A sheet of plate glass is always placed in a vertical plane immediately in front of the candle, between it and the microscope; this partly intercepts the heat of the candle, and prevents any currents of air (caused by the movements of the observer) from causing the flame to flicker.

Thermometers.

The thermometers which are used to indicate the temperatures of bars under comparison have their bulbs at right angles to the tubes; the bulbs passing downwards into circular wells drilled into the bar, while the tube or stem, which is mounted on a metallic plate, lies on the surface of the bar. The metallic plate is six inches in length by three quarters of an inch in breadth. The tube of the thermometer is five inches long; and the length of one degree Fahrenheit is about one fifth of an inch. The tube is divided to tenths of degrees. Four of these thermometers extend from 30° to 50°, four from 50° to 70°, and four from 70° to 90°. Some of the thermometers were made by Messrs. Troughton and Simms, and some by L. Casella of Hatton Garden. The thermometers, when lying in the bars, are read by microscopes *nnnn*, Plate III., whose axes are vertical; the lower part of each of these microscopes is set in a brass slide, whereby it is permitted to traverse the whole length of the thermometer; the corresponding opening in the cover of the box being six inches long by an inch broad. The magnifying power of the microscopes is about ten times, so that the thermometer reading can be estimated to $\frac{1}{100}$ of a degree Fahrenheit; and the field of view is sufficiently large to permit the reading of at least one of the numbers on the scale marking the degrees.

Besides these thermometers with bent bulbs there are some similar thermometers with straight bulbs, which, in the case of standard bars which have no wells, are simply laid upon the surface of the bars.

The errors of the working thermometers (as these may be called) are obtained by comparison with Standard thermometers which have both the boiling and freezing points marked on their scales. The following Standards are twenty-four inches in length:—

"Casella, No. 3241," from 0° to 220° Fahrt. : 1° = 0·0987 inches in length.
„ 3951, „ 20° to 215° „ : 1° = 0·1025 „

and the following, twenty-two and twenty-three inches long:—

"Kew, No. 26," from 20° to 220° Fahrt. : 1° = 0·0962 inches in length.
„ 36, „ 20° to 220° „ : 1° = 0·1000 „

In addition to these, reference has been made to some of the standard thermometers constructed with such very great care and skill by the late Mr. Sheepshanks, during his labours in the construction of the New National Standard Yard and its copies. Those examined have the distinctive marks C_2, A_8, R, and L. These last two are the thermometers which were inserted in his quicksilver trough, and on which in fact almost the whole of Mr. Sheepshanks' operations in the comparison of his own standard depend. The freezing points of L and R, and all the errors of calibration, were determined by Mr. Sheepshanks, but the tubes do not extend to the boiling point. The scale on these thermometers is a scale of tenths of inches. The values were determined by Mr. Sheepshanks in 1850 by comparison with five of his own original thermometers.

Apparatus for Comparison of Thermometers.

Very numerous comparisons amongst the thermometers just described have been made. The apparatus for the comparisons will be seen in Plate VII., Figures 2, 3, 4. It consists essentially of a water trough *mm*, and a bracket BB carrying a microscope which is so held as to be free to slide in two directions at right angles to one another. The bracket is fastened to the wall W of the bar-room (its position, T, is shown in Plate I.), and supported by the prop P; or when the comparisons of thermometers have to be made at high temperatures, the bracket is fixed in another room which is heated by a stove. To each end of this platform an iron plate is screwed, for the purpose of holding the two parallel cylindrical iron rods *xx′*, which are raised about half an inch above the surface of the platform. On these two rods there slides a block of mahogany C. This block grasps the rods by means of three rings on its under surface. Thus it slides with a very even motion in a direction parallel to the length of the water trough *mm*. The brass pieces *zz*, screwed to the upper surface of the block C, form a groove or slide in which the arm *yy* of the microscope slides in a direction perpendicular to the length of the water trough. The rods *xx′* are fixed in an accurately horizontal position.

Into the extremity of the arm *yy* is screwed the tube $\beta\beta$, in which, again, the tube of the microscope AA′ slides. The length of the microscope tube from the object glass to the diaphragm is two feet, and from the object glass to the external focus about the same. In the diaphragm there slides a small rectangular piece of plate glass *aa*, having on its surface a series of eleven converging lines. These are for the purpose of subdividing the spaces on the long thermometers (where a degree is about one tenth of an inch long) into tenths. The lines are made to converge slightly so as to suit different thermometers, the lengths of a degree being slightly variable. The diameter of the object glass is 1·25 inch, and the magnifying power of the eye piece about 8 times. The tube AA′ is very accurately vertical.

The water trough is a wooden box whose internal dimensions are, length 29 inches, breadth 9 inches, depth 9 inches; about the centre of its depth are fixed four bars of wood, *v*, upon which lie the thermometers *tt*, the box being filled with water.

To secure uniformity of temperature in the water, it is agitated by fans *rr′*, two of which are fixed to each of the vertical rods *nn′*. The lower extremities of these rods revolve in pivots, and an upper bearing for them is formed by the brass pieces *opq*, *o′p′q′*. The rod *nn* is made to rotate by the handle *s*, and its rotatory motion is communicated to the other rod *n′* by means of the elastic band *uu*. Thus on the turning of the handle *s* the four fans are set in motion.

In order to prevent the water or thermometers being disturbed by the presence of the observer, a wooden screen, EE, intervenes. This screen is attached to a platform D, on which the observer stands while reading the thermometers; the handle *s* is within convenient reach of his left hand.

In Figure 1 is shown the apparatus for examining the errors of division or of calibration of the long thermometers. On the heavy brass plate *bb′* are erected the uprights *c*,*c′*, which are joined by two parallel brass rods *ee′*. On these rods slides a brass platform, *f*, which holds the rod *e* in two places, or by two bearings, and the rod *e′* in a single bearing. This platform can be fixed in any required position by the screw *g*. The platform carries a transverse sliding piece *i*, into which is screwed a vertical cylinder *h*. In this cylinder the microscope *a* slides vertically, and so is capable of focal adjustment over the thermometer *tt′*. On the plate *bb′* there slides longitudinally a narrow rectangular plate of brass, *dd′*, having a close-fitting bearing at either end. The longitudinal motion is communicated by a micrometer screw *m*. On the upper side of *dd′* there are two small pieces *kk′*, each provided with a spring whereby the thermometer is gently but firmly held in a horizontal position. The micrometer of the microscope is here made no use of; the transverse wire is left as a fixed point of reference in the centre of the field, and all measurements are made by the micrometer screw *m*, which communicates a small motion to the thermometer in the direction of its axis.

Boiling-point Apparatus.

The apparatus for the determination of the boiling point is shown in Figures 5 and 6, Plate VII. Figure 5 is a perspective view of the apparatus supported on a stool at one end, and on a small gas stove at the other. Figure 6 is a section by a vertical plane through the centre. It may be taken asunder into three parts, the boiler *a*, the tube *dd*, and the plate *ff*. The boiler is a cubical vessel (side six inches) having five faces of copper and one *bb* a plate of thick brass; towards the upper part of the face is a rectangular aperture through which the thermometer tubes pass. The copper tube *dd* is soldered at the one end to the rectangular brass plate *cc*, and at the other to the rectangular brass plate *ee*. Both these plates have apertures corresponding to that in *bb*; the apertures are rectangular, 2·75 inches in breadth, by 1·25 inch in height. Inside of the tube *dd* is a second copper tube (Fig. 6) soldered at one extremity to the plate *cc*, but open at the other; falling short of the plate *ee* by half an inch. The transverse section of these two tubes would be simply represented by two similar concentric rectangles. The steam arising from the boiling water in *a*, passes along the inner tube, and then returns along the outside of that tube, escaping by means of the raised part *g* through an aperture in the plates *cc*, *bb*. It does not escape finally by this aperture, but spreads over the upper surface of the boiler *a*, being confined by a copper covering. The object of this is to keep the top of the boiler hot, and so prevent any condensation over the bulbs of the thermometers.

The steam escapes finally by a cock *s*, whose aperture is under control. The manometer *hh*, supplied with a small quantity of *water*, indicates the pressure; the vessel is filled by means of the funnel *k*. The aperture in the brass plate *ee* is closed by means of another plate *ff*, having four small circular holes half an inch apart from centre to centre, through which the thermometer tubes pass. To make the fitting of the thermometer tubes *steam-tight*, they are caused to pass through small holes in a sheet of vulcanized india-rubber. This piece of india-rubber is kept in its place against the plate *ff* by means of another small plate screwed to *ff*: this last plate having, of course, four circular holes corresponding to those in the plate *ff*. Thus the thermometer tubes pass through the circular holes in the india-rubber without coming into actual contact with either of the two brass plates. At the extremity next the boiler the tubes simply rest upon a piece of wire stretched horizontally across the aperture in the plate *cc*. The plane joining the upper edges of the plates *cc*, *ee* is by construction *parallel* to the tubes of thermometers held as described: by this means, laying a level across the upper edges of *cc*, *ee* from one to the other, we can, with the assistance of the wedge *w*, render the tubes of the thermometers accurately level,—a very needful adjustment. The water in the boiler is about an inch below the bulbs of the thermometers. The height of the water in the vessel is indicated in the ordinary manner by a small glass tube (which does not appear in the drawing). The water is heated by means of a small gas stove, which is very convenient, as the amount of heat supplied is thus under ready control. The water used in *a* is from the condensed steam of a large boiler used for other purposes. By means of the small screws shown in the drawing, *c d e f* may be detached at pleasure from *a*; and so also can the plate *ff*, holding the thermometers, be detached from *ee*. In order to render these joints steam-tight, a sheet of vulcanized india-rubber (with a central rectangular aperture) is placed between the plates *bb*, *cc*, and another between the plates *ee*, *ff*. A very small portion of the tubes of the thermometers projects beyond *ff*, only as much as is required to enable one to read the 212° division line properly. The apparatus is placed beneath the long microscope AA′, which is used to read the boiling point.

Supports of the Prussian or Belgian Toises.

These bars have, in all comparisons, as far as is known, been supported on four rollers at a given distance apart and in a straight line. The mode of effecting this, as followed in the present operations, will be understood from Figures 1 and 2, Plate VIII. Figure 1 shows a plan of the box which contains the Prussian and Ordnance Toises when under comparison. The top of the box is removed, and the Prussian Toise is also removed, showing its supporting rollers. TT is the Ordnance Toise resting on its cradles. Figure 2 shows a side elevation of the supports of the Prussian Toise; and the dotted line indicates the box. The box *oooo* containing these bars is of mahogany, an inch in thickness; the internal breadth is six inches and depth five inches; there is a double bottom *o′o′* for rigidity, and also partly to bring the bars to the proper height. The four rollers *e,ee′* are carried by a cast-iron bar *aaa*. This bar is traversed by two short steel cylinders *cc*, Figure 5, which serve as arms or axes, whereby it is held; they have their bearings in brass plates which slide with a vertical motion in the uprights *bb*. These sliding pieces receive their movement upwards or downwards by vertical screws, which are filed square at their upper extremities *kk*. The screws are simultaneously turned by the key, Figure 6, Plate VI. The uprights *bb* are strongly (but removeably) screwed to the bottom of the box. In order to avoid the strain which might occur were no provision made for the relative expansion and contraction of the iron bar and the mahogany box, the holes or bearings in the sliding plates, which at *one* end of the bar receive the pivots *c*, are slightly elongated. It will be seen, then, that we have the means of raising, lowering, or levelling the bar *aa* by means of the key, while the bar itself is held very firmly, and yet altogether free from strain.

The manner in which the bar *aa* holds the rollers or rather the roller frames will be understood by Figures 4, 7, 8. The frame, of brass, *dd*, fits *closely* upon the iron bar, being bolted through by the bolt f; but the hole through which f passes is slightly elongated in a vertical direction, so that the frame *dd* may be moved upwards or downwards, but the movement is purposely stiff, and not easy. By driving the little screws f, f', whose points rest on the upper surface of the bar *aa*, the frame, with the roller it carries, is raised. On unscrewing f, f', and relieving thus their points from contact with the bar, the frame, with its roller, may be pressed down (requiring some force) by the hand until the points of f and f'' again come into contact with the bar. Thus it will be seen that we have the power of raising or lowering the rollers; but this applies only to the two central rollers *ee*; the extreme rollers e, e' have no vertical adjustment; it is sufficient that the other two *ee* can be brought into a straight line with e, e'. One of the extreme rollers e, is capable of a slight transverse motion by means of the slow motion screw g; the other roller e' has no adjustment. Two of the rollers e, e' have flanges; the other two, *ee*, have not; one of the central rollers, Fig. 8, is a cylinder; the others are all slightly convex or barrel-shaped. The object of this is to secure for the Toise an unconstrained bearing; were the rollers all cylinders there would be a danger of submitting the bar to a small torsion force. In Figure 2 *tt* is the Toise resting on the four rollers. To the extremities of the bar *aa* are fixed the horizontal brass plates *hh*, which carry the two parts of—

The Contact Apparatus.

This is shown in plan in Figure 3, with the Toise lying in between; and in perspective, mounted on its stand in Figure 6. It will materially simplify the explanation of this apparatus if we first explain that the two flat semi-cylinders *pp'* are the *essential parts;* the remainder of the screws, &c. merely afford the adjusting power whereby we are enabled to place these semi-cylinders in any required position. Each of the semi-cylinders has a fine line on its surface (parallel to its base or diameter), and about the hundredth part of an inch from the circumference; so that when the two semi-cylinders are in contact, as in Figure 6, these form a pair of parallel lines; parallel to, equidistant from, and on opposite sides of the line, which is then a common tangent to the two semi-cylinders.

When under comparison, the Toise lies between the two parts or halves of this apparatus; one half being fixed on each of the plates *hh* (Figures 1, 2) at the extremities of the bar *aa*. During the comparisons the semi-cylinders are in contact with the terminal polished disks of the Toise; so that if σ be the distance between the parallel lines on the semi-cylinders when they are in contact as in Figure 6, the distance between the same lines when the two semi-cylinders are mounted on the plates *hh* and in contact with the Toise, whose length is τ, is $\sigma + \tau$. Strictly speaking, the pieces we have called semi-cylinders are not so. They are formed thus:—A cylinder is turned of steel 0·75 inch in diameter and 0·12 inch thick; its cylindrical surface is then made slightly convex, so that the solid becomes a segment of a prolate spheroid contained between two planes perpendicular to the axis of revolution, and at equal distances on either side of the centre of the generating ellipse. The radius of curvature of this ellipse at the extremity of its transverse axis is about two inches. The surface of the steel is before removal from the lathe very highly polished, all the marks of turning being worked out. The piece of steel is then cut in halves along a diameter. When the two halves are laid upon a horizontal plane surface, and their curved edges brought into contact, it is clear that their common tangent plane is vertical. This *may* not be true if through any imperfection in the turning, the solid produced is a segment of an ellipsoid contained between two planes at *not* exactly the same distance on opposite sides of the centre, unless the faces which are uppermost formed originally the same plane, that is before the cylinder was cut.

It will be seen by this that the semi-cylinders have contact with the vertical terminal steel disks of the Toise, not over their whole surface but in a *point* only. This has the advantage that there is the less chance of any particle of dust, &c. making the contact false. It would have been easier to have made an apparatus which should have presented a plane surface to be brought against the terminal disks of the Toise, but there would have been greater risk of imperfect contact.

It is necessary to the perfection of the comparisons that we have perfect control over these semi-cylinders. Now, in order that the position of a body in space be fixed, six quantities are required to be given. For instance, suppose a sphere, and let there be two points marked on its surface P, Q. Its centre is fixed by three rectangular co-ordinates, but it is still free to revolve in any manner round the centre. But if we have the altitude and azimuth, so to speak, of that point in which the line joining the centre and P meets the celestial sphere, the body is now free only to revolve about a fixed line; and one more quantity given will fix its position absolutely. And it will be found that the same number of quantities are required whatever may be the method of fixing the body.

Now, in adjusting the semi-cylinder to contact with the plane terminal circular disk of the Toise, there are six things to be considered:—

1. The diameter or base of the semi-cylinder must be horizontal:
2. The radius drawn on the surface perpendicular to this diameter must be horizontal:
3. The contact with the disk of the Toise must be at the proper altitude, that is, on the horizontal diameter of the disk:
4. The contact must also be on the vertical diameter of the disk:
5. The contact must not be intercepted by any particle of dust or other matter:
6. The fine line (parallel to the base or diameter) which is drawn close to the circumference of the semi-cylinder must be parallel to the terminal disk of the Toise.

Of these conditions the first and second secure the horizontality of the plane faces of the semi-cylinder; the third and fourth secure a contact at the centre of the disk of the toise; the fifth, an unintercepted contact; and the sixth provides that the actual contact shall be at that point of the convex surface of the semi-cylinder in which the vertical tangent plane is parallel to the fine line on the surface.

We now proceed to explain Figures 3 and 6. The semi-cylinder p is firmly screwed to the rectangular steel plate—we shall call it the steel needle, l. This steel needle slides, without any looseness or shake, between the brass pieces $kk'k'k'$ which are screwed to the piece ggg. It is urged forward by the spring point n, the spiral spring urging n being contained in the small cylinder m. The force of the spring can be regulated by the screw μ; the cylinder m is fixed to the piece ggg. This piece ggg is also made to slide, between ff and $f'f'$, on the plate dd; the motion is communicated by the screw γ, which has a shoulder at h fixed to ggg, and works in the female screw h', which is connected with the plate dd. This last plate, again, slides between the pieces bb, on the lower plate aaa. The motion is communicated by the screw δ, which works in a shoulder attached to the plate aaa. The screws γ δ then communicate longitudinal and transverse motion to the semi-cylinder p.

The plate $aaaa$ rests upon either the plate h, Figures 1, 2, or upon the stand AAA; the contact being on the points of the four screws $ccc'c'$. By these screws we may either level or raise bodily the plate aaa. To support a body on four points is generally objectionable; but in the present case not the slighest difficulty arises from this cause, as the very minutest amount of shake is readily discovered on slightly tapping the head of either of the screws $c'c'$. Finally, the plate aaa is held down, either to the stand AA or to the plate h, Figures 1, 2, by the shoulders of the screws $ee_, e_,$, which pass through holes (without threads) in the plate aaa.

For the purpose of levelling, a small but very delicate level stands upon the needle l.

The holes in the stand AAA for the three screws $ee_, e_,$ of each half of the contact apparatus, are so cut by the maker, that the directions of the sliding motions of the two needles shall be exactly parallel. In order to draw the lines on the steel cylinders, small pieces of platinum are inlaid in a groove on their upper surfaces. The two parts are then mounted on the stand, and the semi-cylinders being brought lightly to contact, the transverse slow-motion screws δ are turned until the common tangent line at the point of contact is perpendicular to the line of motion of the needles. Lines are then drawn, as in the accompanying Figure 1.

Fig. 1.

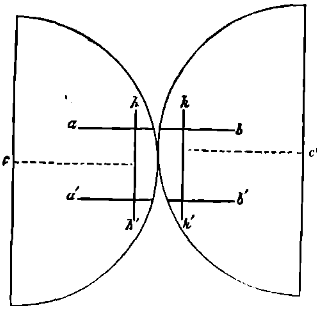
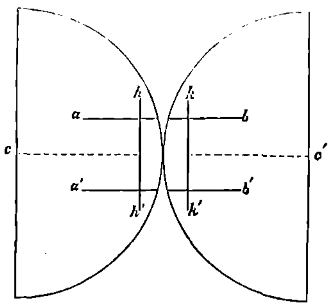


Fig. 2.



The dotted line joining the centres cc', Fig. 2, corresponds with the line of motion of the needles, and the lines ab, $a'b'$, necessarily broken, are drawn parallel to and equally distant from cc'. Then perpendicular to these and parallel to the common tangent line are drawn the lines hh' kk'. If, however, the semi-cylinders be not accurately opposite to one another when the lines are drawn, and thus their centres describe, not the same straight line, as in Figure 2, but parallel lines as in Figure 1, at a very small distance apart, and if the lines ab $a'b'$ be then drawn, neither of them equidistant from either of the dotted lines, then the distance of the transverse lines hh' kk' can be increased by adjusting the two centres cc' into coincidence with the line of motion of the needles. But this maximum distance is the quantity absolutely required to be known, as will be seen if we suppose the semi-cylinders in contact with the vertical parallel terminal disks of the Toise, the lines hh' kk' being parallel to those disks.

We may correct for this error in the relative position of the semi-cylinders when the lines are drawn, in the following manner. Let ϱ be the radius of either semi-cylinder, $\varrho - \lambda$, $\varrho - \lambda'$, the distance of hh' and kk' from the centres cc' x, the distance apart of the parallel lines through cc', also let δ be the distance apart of the parallel lines hh' kk' when the cylinders are in contact as in the figure. Now, (1) by the motion of the transverse adjusting screws let the semi-cylinders, maintaining their parallelism and contact, be so placed that a b' shall fall into one and the same straight line; in this position let the distance of hh' from kk' be δ': (2), let, similarly, a' and b be made to fall into one and the same straight line, and in this position let the distance of hh' from kk' be $\delta_{,}$. Then we have the following equations:

$$(\delta' + 2\varrho - \lambda - \lambda')^2 + (i + x)^2 = 4\varrho^2$$
$$(\delta_{,} + 2\varrho - \lambda - \lambda')^2 + (i - x)^2 = 4\varrho$$
$$(\delta + 2\varrho - \lambda - \lambda')^2 + x^2 = 4\varrho^2$$

Where i is the distance apart of the parallel lines ab $a'b'$. These equations become, since i and x are very small compared with ϱ

$$\lambda + \lambda' - \frac{(i + x)^2}{4\varrho} = \delta'$$

$$\lambda + \lambda' - \frac{(i - x)^2}{4\varrho} = \delta_{,}$$

$$\lambda + \lambda' - \frac{x^2}{4\varrho} = \delta$$

$$\therefore \frac{ix}{\varrho} = \delta_{,} - \delta'$$

consequently x becomes known by the measurement of $\delta_{,} - \delta''$, and we have

$$\lambda + \lambda' = \delta + \frac{\varrho}{4\,i^2}\,(\delta_{,} - \delta'')^2$$

The parallelism of the lines hh', kk' when the apparatus is mounted on the stand AAA is very perfect, as is easily verified by the motion of the transverse wire of a micrometer microscope.

2.

National Standard Yards.

The various copies of the National Standard Yard, constructed of different metals, as cast steel, cast iron, Swedish iron, bronze, &c., are all one inch square in section. Towards the extremities of one face of the bar, and thirty-six inches apart from centre to centre, two cylindrical wells are sunk to the depth of the centre of the bar; in the centre of the bottoms of the wells gold pins are inserted, and upon the prepared surface of each, three equidistant parallel lines perpendicular to the length of the bar are drawn. The distance apart of these parallel lines is one hundredth of an inch (very approximately). They are crossed at right angles by a pair of parallel lines 0·02 inches apart, parallel to the length of the bar. The point to be measured from on each gold pin is that point of the middle transverse line which lies midway between the longitudinal parallel lines. There are four wells in the upper surface for receiving the bent bulbs of thermometers; these wells (in those copies distributed to the Ordnance Survey Office) are at five and fifteen inches from the centre on either side.

Ordnance Survey Standard.

This Standard was constructed for the Ordnance Survey in 1826–7 by Messrs. Troughton and Simms; it is of wrought iron 122·15 inches in length, 1·45 in breadth, and 2·50 inches in depth. It is supported on two rollers at one fourth and three fourths of its length. The ends of the bar are cut away to half its depth, so that the dots marking, on platinum pins, the measure of 10 feet, are situated in the neutral axis of the bar. On the upper surface, 40 inches from the middle of the bar towards either extremity, are two wells for thermometers. This bar has been always designated $\mathbf{O}_1$.

Ordnance Intermediate Bar.

In order to make a comparison between the lengths of the two bars just described, that is, to obtain the length of the Ordnance ten-feet Standard in terms of the standard yard, it was necessary to construct an intermediate bar of ten feet, subdivided into yards and feet. This bar has its section in the form of a girder, with equal flanges above and below, as will be seen by referring to Plate IV. It is of wrought iron, carefully planed. The breadth is 1·53 inches, and depth 2·48 inches. A groove half an inch in breadth, and about an eighth of an inch deep, is planed out through the whole length of the upper surface. In the centre of the breadth of this groove seven holes were drilled, about a tenth of an inch diameter, one at the centre of the bar, and three on each side of the centre, namely, at one foot, two feet, and five feet from the centre; thus dividing the bar into six spaces, viz., one yard on the left, four contiguous spaces of one foot each, in the centre; and one space of a yard on the right. Into the holes just described were screwed small cylindrical iron plugs, whose heads were afterwards filed off level with the upper surface of the bar, but raised above the bottom of the groove. The tops of these plugs were then with some considerable difficulty and labour filed down, until they appeared to be in true horizontal line—the bar resting on eight rollers; that is, no one above the level of the others. This was tested by a delicate level, and a platinum pin was then inserted in the centre of each plug. A fine thread was then stretched along the length of the bar over the centres of the platinum pins, and on each pin was drawn a pair of parallel lines, one on either side of the thread, and so about 0·02 inch apart. Finally, on each pin was drawn a single transverse line, perpendicular to the parallel lines. The measures are taken from that point of the transverse line which is midway between the longitudinal parallel lines.

In the upper surface of the bar are four wells for thermometers, one in the centre of each of the extreme yard spaces, and one in the centre of each of the spaces of one foot adjacent to the centre; so that the wells are, six, and forty-two, inches from the centre bar towards either end.

This bar is designated **OI_1**.

Similar, except in the disposition of the subdivisions, are the bars **OI_2 OI_4 OI_6**: the last two are the Australian Standards. All these have been invariably supported on the cradle system of eight rollers.

Ordnance Toise.

As in the bars just described, the section of this bar is that of a girder with equal flanges above and below; the exact section is shown in Figure 8, Plate VI. It is of cast steel, carefully planed. The points are inserted in the same manner as in the bar **OI_1**; they are four in number, subdividing the bar into three spaces, namely, two consecutive yards, and a space of 4·74 inches, so that the distance between the extreme points is 76·74 inches. On each of the four platinum pins are drawn, not one transverse line, as in **OI_1**, but, as in the standard yards, a system of three parallel equidistant transverse lines, the central line being that used in the measurements. The distance apart of the parallel transverse lines is, as in the standard yards, one hundredth of an inch. The breadth of this bar is 1·00 inch, and its depth 1·50 inch; the extreme length 78·0 inches. Through the upper surface are drilled two wells for thermometers, 22·5 inches on either side of the centre. The bar rests on a cradle system of eight rollers. It is designated **OT**.

Ordnance Metre

Is a bar the same in section, material, and general construction as the preceding, but has only three points on its surface, subdividing the length into two spaces, one of which is a yard, and the other 3·37 inches, so that the distance of the extreme points is 39·37 inches. The extreme length is 41 inches; the thermometer wells are 11·75 inches on either side of the centre of the bar. The bar rests on a cradle system of four rollers. It is designated **OM**.

Ordnance Foot.

This is a bar of wrought iron, an inch square in section and thirteen inches in length. The lines marking the inches into which the foot is divided are drawn on platinum pins, which are flush with the upper surface of the bar. The extreme inches are divided on inlaid strips of platinum into tenths, and two of the tenths in each inch into hundredths. There are two wells for thermometers, three inches and a half on either side of the centre of the bar. It is designated **OF**.

Prussian and Belgian Toises.

These are flat bars of cast steel, similar in all respects. The breadth is 1·70 inches, depth 0·39 inch; terminating at each end in a cylinder 0·56 inch long, as shown in Figure 3, Plate VIII., the diameter of the cylinder coinciding with the depth of the bar. At the extremity of each of these cylinders is affixed a smaller cylinder of tempered steel, having its axis coincident with the axis of the bar; its diameter is 0·12 inch, its length only 0·016 inch. The faces of these small cylinders, which are perfect planes, beautifully polished, and at right angles to the axis of the bar, form the terminal planes (*les bouts*) of the measure. The Prussian bar has the distinctive mark No. 10.; the Belgian is marked No. 11. They were made by M. Baumann of Berlin, and are copies of the Toise of Bessel, preserved in the Observatory at Königsberg.

The Prussian bar bears the following inscription :—

1852. *Dieser Stab in der Wärme von* 16·25 *graden des hunderttheiligen thermometers, in der Achse seiner gedrehten enden gemessen, ist* 0·00019 *linien kürzer als die der Königsberger Sternwarte gehörige copie der toise du Pérou.*

No. 10. *Baumann, Berlin.*

The Belgian bears the following :—

1852. *Dieser Stab in der Wärme von* 16·25 *graden des hunderttheiligen thermometers, in der Achse seiner gedrehten enden gemessen, ist* 0·000202 *linien kürzer als die der Königsberger Sternwarte gehörige copie der toise du Pérou.*

No. 11. *Baumann, Berlin.*

These bars are supported each on four points adjusted into a straight line, the distance apart from one to another being 21·5 inches.

II.

ON THE SOURCES OF ERROR

AND

THE METHODS OF COMPARING.

1.

The elasticity of the material of which standards of length are constructed rendering them to a certain degree susceptible of change of form, it is of the utmost importance that a standard bar be so held as that the strain to which it is subjected be the least possible. Moreover, it is absolutely necessary that the free expansion and contraction of the bar be not impeded. By the cradle system of lever rollers these points are both secured, for when the friction of the rollers is reduced to a minimum the bar moves with a very slight pressure. Thus the ten-feet bar **OI$_1$**, weighing 81 lbs., when resting on its eight rollers is moved by a force of 4·40 lbs.; and the bar **OT**, weighing 22 lbs., when on its eight rollers is moved by a force of 1·03 lbs.

The standard yard being a comparatively short bar, has been, in the present comparisons, supported always on two rollers at one fourth, and three fourths of its length, care being always taken that the rollers were free to revolve with the least force. It was satisfactorily proved by the late Mr. Sheepshanks that the length of this bar is not affected by the mode of support; the reason being that the lines are engraved on a surface coinciding with the neutral axis of the bar.

It is of much importance that the supporting apparatus upholding the bars under comparison be strong, weighty, and free from shake; it should also be as far as possible from possessing any elasticity, and should be not liable to be disturbed by the presence of the observer or his necessary movements. In the present operations this point has been carefully studied; the beams which support the travelling carriages have a foundation entirely separate from the flooring; and the travelling carriages are so constructed that any shake is next to impossible, nor can it be introduced by wear. The boxes which contain the bars are very strongly put together, and have considerable rigidity; they are, moreover, supported on three points. The boxes always are supported at the same parts of their bottoms, brass plates being attached below, which plates come into immediate contact with the bosses on the carriages.

The room in which the comparisons are made is specially adapted to exclude sudden changes of temperature. A change of more than one degree Fahrenheit in the course of 24 hours is not very common, and sometimes there has not been a change exceeding one degree for weeks. Still there is even in this room caution required, especially when the general state of the weather has changed suddenly from warm to cold, or vice versâ; for in this case it would appear that the stone pillars which carry the microscopes are slower to acquire the change of temperature than the air of the room. Here then the two bars which lie in the box under the microscope are dissimilarly situated; the one is nearer the stone piers; the other is nearer the centre of the room; and there exists a slight difference of temperature between them, some one or two tenths of a degree. The remedy for this is to observe with the bars in alternate positions, that bar which is nearest the piers one day being put nearest the observer the next day. By this means we eliminate, or very nearly, at any rate as nearly as possible, the effect of slight difference of temperature between the bars; but it is to be noted that the computed probable error of the results in such a case will be greater than it should be. When the temperature of the two bars differs, as just described, the thermometers do not *always* indicate the difference; its existence is rather inferred from the results of the comparisons.

Another evil not easily remedied is that the bars compared are not all of the same section, and consequently not all equally sensitive to changes of temperature, the more massive bars being slower in taking up a change of temperature than the lighter ones.

A slight disturbance and rise of temperature is almost inevitably caused by the presence of the observer when making the observation, and also partly, perhaps, by the heat of the two candles, which are then lighted, one at each microscope. The effect has seldom amounted to 0·06 on the mean of the four thermometers. The box remains covered during the comparisons.

2.

The method of adjusting the different parts of the apparatus for the comparison of two bars is as follows:—The box which is to carry the two bars is placed on the carriages, and the bars then placed on their supporting rollers in the box. The box being run in close to the piers until the wheels $n_1 n_3$ Plates IV., VI. of the upper carriages are within an eighth of an inch of contact with their stops $v_1 v_2$, one of the bars, that nearer the observer (or the *outer* bar), is carefully levelled by means of a long level laid on its surface.

The microscopes H and K are then placed in the cast-iron stands over the extremities of the bar which has just been levelled. The axes of rotation of the two microscopes are made approximately vertical, by revolving them in their gun-metal holders; they are then by means of the lower nuts $\gamma_{\prime}\gamma_{\prime}\gamma_{\prime}$, Plate VI., Figure 1, brought as nearly as possible to focus over the lines on the platinum disks on the bar. The verticality of the axes of the microscope is then again corrected, and the microscopes brought by the sliding of the cast-iron stand (on the stone piers) to bisect approximately by their cross hairs (which are set to zero in the collimation centre of the field) the lines on the bar. These different operations, necessarily interfering with one another, may have to be gone over separately several times before they all stand perfect together. When two bars are compared, whose lengths, as is usually the case, do not differ more than from five to ten micrometer divisions, the microscopes are generally so placed (the micrometer-heads being outwards, as shown in Plate III.) that each line on the bars will read a small positive number of divisions; that is, supposing the centre of the field to read zero; but if the difference be large, as in the comparison of the Ordnance and Prussian Toises, where it amounts to upwards of 500 divisions, then the microscopes are so placed that the distance of their zeros is a mean between the lengths of the bars; thus in each microscope equal quantities are measured on each side of the zero or collimation centre. In this case the collimation centre is supposed to read 10 revolutions, so as to avoid changes of sign in the readings.

The microscopes being brought into adjustment, the upper nuts $\gamma'\gamma'\gamma'$ are brought down gently, so as to clamp the arms of the gun-metal holder firmly. This is liable to disturb some of the preceding adjustments, which are therefore again examined and corrected. The box is now run out from the piers, until the clamp *s* comes to about the end of its slide; in this position the second or inner bar will be found under the microscopes; and either of the upper carriages being run in or out small quantities; and the whole box run longitudinally a small quantity if necessary, by the motion of both carriages on the rails *aa*, the divided disks are brought into the centres of the fields of the microscopes. In this position the upper carriages are clamped, and the slow-motion screw σ of each, worked until the intersection of the cross-hairs of the microscopes are found midway between the parallel longitudinal lines on each disk. A small motion is then communicated by hand to the box longitudinally, until the lines to be observed are nearly bisected. This being done, the bar, which was not previously levelled, has to be brought to focus under the microscopes (which remain fixed). This is done by raising or lowering the cradles by means of the levelling key, Figure 6. Plate VI.

Thermometers are now laid on the bars, their bulbs being inserted in the wells, and the cover placed on the box. On the cover are then fastened (over the thermometer openings) the small brass slides by which the reading microscopes *nnnn*, Plate III, are held, and permitted to run up and down the length of the thermometer tubes.

It is usual to arrange a pair of bars for comparison, in the afternoon or evening of one day, and to commence the observing the next day. The bars are visited three or four times each day; a *series* of comparison has generally consisted of ten visits or comparisons; and the bars are then dismounted, to be compared, if necessary, another time. It is desirable to have the comparisons of a given pair of bars to extend over as many days as possible, and the comparisons to be broken up into series, so that one series cannot have errors common to the preceding or following series. If a pair of bars be adjusted under microscopes for comparison, then as long as all, or only some, of the adjustments are not interfered with, there may be a constant error; but if the whole be dismantled, and the comparisons recommenced after some days, and so on, we are much more certain of arriving at the truth. There is little use in multiplying observations within a short space of time during which none of the circumstances of the observations are changed. When observations are made in this manner the results may apparently be very excellent, but they are very liable to be affected by some constant error. It is rather required to bring out all the discrepancies in the observations that can be legitimately brought out, by varying, as much as possible, the circumstances of the observations; in each case, of course, taking care that no known error of adjustment is left, nor any known cause of error in action. If, therefore, two bars have to be compared, it is desirable to break up the comparisons into detached series. If anything should prevent this, the different adjustments must be at least renewed as frequently as possible, no one being left undisturbed.

In the comparing of two bars it is most desirable that the majority of the comparisons be made within a few degrees of 62°, this being generally the standard temperature of reference. A smaller number of comparisons at a low temperature, as 30° or 40°, will then enable us to reduce all the other comparisons to 62°.

It is assumed in the reduction of the observations that the temperatures of the two bars lying side by side in the box are the same. We might have taken them to be of the temperatures indicated by the thermometers, and thus small differences would be found; but it is a question whether the thermometers, *do* indicate the temperature of the metal with such precision as that these differences of reading might be taken to represent the actual differences of temperature of the bars. It has been considered safer to assume the bars to be generally of the same temperature—with the precaution of causing them to exchange positions.

3.

The errors of the short working thermometers are obtained from time to time by comparisons with the Standard Thermometers at the temperatures at which they have stood while in the bars. To explain the manner of comparing, suppose two working thermometers are to be examined; they are laid by the side of two Standards in the water trough, which is filled with water of the requisite temperature, and with their bulbs as nearly as possible occupying the same part of the water. While so arranged they are at the mid-depth of the water, being covered by about four inches of it. Suppose the thermometers have to be compared at 62°, and the temperature of the air in the place where the water trough stands is at 60°; the water in the trough is made 63°, by the addition of a little warm water, and during the course of, say six hours, it may fall from 63° to 61°, during which time many comparisons may be made. A single comparison is made as follows:—the water being first agitated by the revolving of the handle; the thermometers are read by means of the long microscope AA', first in the direct order A, B, C, D, and immediately after in the reverse order D, C, B, A. The mean of these two readings of each thermometer is taken as the result of this comparison. The errors of observation in this operation are very small; all the thermometers are read to one hundredth of a degree. Four comparisons are generally made at a visit; the water being stirred immediately before each.

In order to the obtaining of good results, it is necessary that the temperature of the room be not more than a very few degrees different from that of the water in the trough. Otherwise the water cools too quickly.

With respect to the long standard thermometers which are divided to half degrees Fahrenheit, the errors of the division lines, as subdividing into parts of *equal capacity* the tube from the line 32° to the line 212°, have been determined (to a certain extent) by the process of calibration. Here it is to be remarked that these thermometers having already gone through this process at the hands of the maker, or at Kew Observatory, before the lines were finally drawn, the residual errors to be determined are very small. First, to determine the error of the division line marking 92°; from 32° to 92° should be *one third* of the capacity of the tube from 32° to 212°; a column of about 60¼ degrees is accordingly broken off and run along the tube until its extremities slightly overlap the lines 32°, 92°. The one extremity will lie between 31° and 32°; the other between 92° and 93°. The thermometer is now laid on the supports, where it is held by two small springs. The microscope is caused to slide along its supporting rods until the fixed wire is nearly bisecting the line 31°, where it is clamped. It is impossible to have both the column of mercury and the divisions on the glass tube in focus at the same time, and the adjustment must be divided between them; this is an unavoidable source of error. Care is taken that the wire in the micrometer is parallel to the lines on the thermometer. Now by the movement of the micrometer *m*, Fig. 1, Plate VII. the thermometer is drawn along longitudinally, and, *first*, the reading of the micrometer is noted when the line 31° is bisected; *secondly*, when the extremity of the column of mercury is bisected; and *thirdly*, when the line 32° is bisected. The difference of the second and third readings, divided by the difference of the first and third, gives the fraction of a degree by which the extremity of the column lies beyond the line 32°. Taking care that the thermometer is not shaken, the microscope is run along its supporting rods until the fixed wire arrives at the line 92°, when it is clamped. Then by the micrometer *m*, the line 92°, the end of the column, and the line 93°, are successively brought into coincidence with the fixed wire in the microscope. The difference of the first and second readings, divided by the difference of the first and third readings, gives the fraction of a degree by which the column lies beyond 92°. Let the sum of this fraction and that previously obtained, for the other end of the column, be s_1. Now tilting up the frame, Fig. 1, Plate VII., without touching the thermometer, the broken column is made to slide along until its ends overlap, by nearly equal quantities, the lines 92°, 152°. The fractions of degrees which these quantities measure are then found as before; let their sum be s_2. Again, the column is made to slide along the tube until its ends slightly overlap the lines 152°, 212°. Let the sum of the fractions in this case be s_3. Now, if we by $[a \cdot b]$ mean the capacity of the tube between the two lines a and b, and if C be the volume of the column of mercury:

$$C = [32\cdot92] + s_1$$
$$C = [92\cdot152] + s_2$$
$$C = [152\cdot212] + s_3$$
$$\therefore C = \tfrac{1}{3}\,[32\cdot212] + \frac{s_1 + s_2 + s_3}{3}$$
$$\therefore [32\cdot92] = \tfrac{1}{3}\,[32\cdot212] + \frac{-2\,s_1 + s_2 + s_3}{3}$$
$$[92\cdot152] = \tfrac{1}{3}\,[32\cdot212] + \frac{s_1 - 2s_2 + s_3}{3}$$
$$[152\cdot212] = \tfrac{1}{3}\,[32\cdot212] + \frac{s_1 + s_2 - 2\,s_3}{3}$$
$$\therefore [32\cdot152] = \tfrac{2}{3}\,[32\cdot212] + \frac{-s_1 - s_2 + 2\,s_3}{3}$$

Therefore the errors of the division lines 92°, 152°, are—

$$92^\circ \ldots\ldots\ldots \frac{-2\,s_1 + s_2 + s_3}{3}$$
$$152^\circ \ldots\ldots\ldots \frac{-s_1 - s_2 + 2\,s_3}{3}$$

It is almost unnecessary to say that this operation will have to be repeated several times before the exact truth can be arrived at.

Again, breaking off a column of thirty degrees, we may, by making it lie first between 32° and 62° and then between 62° and 92°, so obtain the error of the line 62°. Then by a column of 20 degrees we can get the errors of the lines 52°, 72°; and measuring from 62° get the errors of 42° and 82°. Thus we have errors for every ten degrees from 32° to 92°, and it is only a matter of time and labour to continue the investigation to any extent; but the work is very tedious, and each part must be frequently repeated, as in spite of all precautions that can be devised discrepancies will present themselves.

By this process we find the errors of the different divisions with reference to the lines 32 and 212. It remains to ascertain the accuracy of these two lines *relatively* to one another. For this purpose the thermometer is boiled at the standard barometric pressure, and within a few minutes afterwards, placed with its bulb in melting pounded ice, or snow. Let the readings be $212 + b$ and $32 + f$, then the correction to the reading of the Standard Thermometer at the temperature t is

$$-\frac{t-32}{180}\,(b-f)$$

The correction for the error of the *absolute position* of the line 32 is determined from time to time, inasmuch as it is in all thermometers a somewhat variable quantity. At any given time, let $32 + f'$ be the reading of the Standard Thermometer when placed in melting pounded ice, or snow, then the total correction is

$$-f' - \frac{t-32}{180}\,(b-f) + s$$

where s is the correction resulting from the process of calibration.

It is almost unnecessary to remark that if the thermometer be boiled at a barometric pressure differing from the standard pressure, the reading of the thermometer is corrected accordingly.

III.

ON THE CHANGES OF FORM OF A BAR OF METAL CONSIDERED AS AN ELASTIC BODY.

If we suppose a bar of iron, steel, or any other suitable material to have a fine dot engraved upon one of its surfaces and close to either extremity, the distance of these dots from one another depends not only on the temperature of the bar, but on the magnitude and directions of the forces by which the bar is held in its position under the microscopes which view the dots. These forces, or, in other words, the supports of the bar, may be varied, and hence a variation of length will result from the change of form due to the elasticity of the metal. The mode of supporting a measuring bar becomes, therefore, a matter of great importance. A considerable number of experiments on these variations of length were made by the late Mr. Baily, as recorded in the "*Account of the Construction of the new National Standard of length and its principal Copies*" pages 16 and 17, and it has been considered desirable to obtain further information on this point by very carefully conducted comparisons, and to compare the results obtained from such observations with those obtained from the ordinary theory of the flexure of elastic rods.

1.

Imagine an elastic rod or bar whose surfaces are perfect planes and its section a rectangle whose breadth is h and depth k; let also a be the length and w the weight of the bar. As only small forces are to be here considered, we shall assume that the extension and compression are equal for the same absolute amount of force. Let α be the extension of the bar when directly extended by a force equal to its weight w. So that α also represents the compression of the bar when compressed by a force equal to w; then $\varrho\,\alpha$ will be the extension or compression resulting from an extending or compressing force $\varrho\,w$.

FIG. 1.

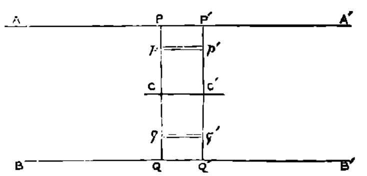
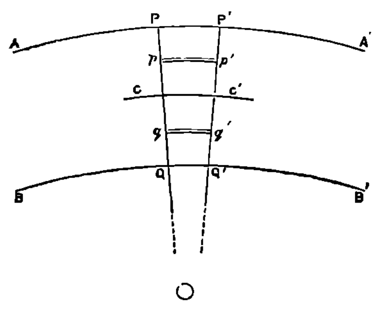


FIG. 2.



Let AA′, BB′ (fig. 1) represent the side elevation of a small portion of the bar when lying on a horizontal plane. PQ, P′Q′ represent two planes very close to one another perpendicular to the bar's length, and parallel to one another; CC′ represents a horizontal plane equidistant from the planes AA′ BB′, the parallel lines pp' define a portion of the solid or a lamina contained between two horizontal planes indefinitely close together, the two vertical planes PQ, P′Q′, and the two vertical bounding surfaces of the bar.

Figure 2 represents the same portion of the bar when curved in a vertical plane. We suppose the forces which produce the flexure to act upon the upper and under surfaces perpendicularly to those surfaces and equally over the whole breadth of those surfaces, as for instance, if the bar be resting on two horizontal cylinders, whose axes are perpendicular to the direction of the bar's length. As the flexure of the bar is always a very minute quantity, the force of gravity will also act in a direction sensibly perpendicular to the bar.

Take a point z on the surface of the bar, and suppose a plane to pass through this point perpendicular to the bar when straight. Then it is assumed that all the particles of the body which lie in this plane, will, when the bar is curved, continue to lie in a plane passing through z, and perpendicular to the four surfaces of the bar. Consequently the particles of the body which in figure 1 lie in the planes PP′ QQ′ will, when the bar is curved, figure 2, lie in the two planes PP′ QQ′ which converge to the centre of curvature O of the bar. The lamina pp' (figure 1) will become (figure 2) part of a cylindrical shell, and will be in a state of tension while any lamina qq' (figure 1) will become also a part of a cylindrical shell (figure 2), but in a state of compression. As a necessary consequence of the assumption that extension and compression are equal for the same amount of directly applied force, it follows that the extension of the upper surface PP′ and the compression of the lower QQ′ are equal, and also that a lamina occupying the central position CC′ the neutral surface is neither extended nor compressed.

If ϱ be the radius of curvature of CC′, u the distance of pp' from CC′, then the extension of pp' is

$$\varepsilon = \sigma \frac{u}{\varrho} \tag{1}$$

where σ is the original length of pp' or the distance of the parallel planes PP′, QQ′, figure 1. This formula evidently includes the case of compression for negative values of u. It is further assumed that the elastic forces developed in the plane P′Q′, and acting perpendicularly to this plane, are just such as produce the extensions expressed in formula (1). These forces are estimated as follows: Since the force w extends the bar whose length is a and area of section hk to the amount α, it would extend a lamina whose thickness is δu, breadth h, and length σ to the amount

$$\frac{\sigma}{a} \cdot \frac{k}{\delta u} \alpha$$

Consequently the force necessary to extend this lamina to the amount ε is

$$\frac{wa\,\varepsilon}{k\,\sigma\alpha}\,\delta u$$

and substituting from (1) the value of ε, we get—

$$\frac{wa}{\varrho\alpha\,k}\,u\,\delta u \tag{2}$$

for the force brought into play at p or p' in the direction of the bars length. The resultant of these parallel forces or their sum over the whole sectional surface P′Q′ is

$$\frac{wa}{\varrho\alpha\,k}\int_{-\frac{k}{2}}^{\frac{k}{2}} u\,du = 0 \tag{3}$$

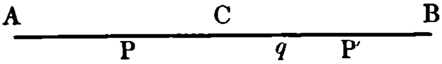
and this result is necessary for the equilibrium of the portion of the bar to the right of P′Q′, as the only forces by which it is solicited are perpendicular to its length. Further it is necessary for the equilibrium of the same portion of the bar that the sum of the moments of the applied forces with respect to C′, should be equal to the sum of the moments of the elastic forces in P′Q′ round C′. This latter sum is

$$\frac{wa}{\varrho\alpha\,k}\int_{\frac{k}{2}}^{\frac{k}{2}} u_2\,du = \frac{wak^2}{12\,\alpha\varrho} \tag{4}$$

so that when the forces which act perpendicularly upon the bar are given in magnitude and position, the radius of curvature of the neutral axis at any point will follow from (4).

2.

Let us now investigate the form assumed by an elastic rod resting upon two supports in a horizontal line. Let AB be the rod whose length is a and weight w, PP′ the points of support at distances b and b' from the centre C of the bar. Let the equation of the curve be expressed in rectangular co-ordinates $x\ y$, the axis of x passing through the points of support, and $x = 0$ corresponding to the centre of the bar. Now consider any point q of the bar between C and P′ and let C$q = x$, the forces acting upon the portion qB of the beam are, 1st, the reaction of the support P′; 2d, the system of parallel forces constituting the *weight* of qB, and which may be replaced by their equivalent the weight acting at the middle point of qB, and the elastic forces brought into action in the section of the bar at q. The sum of the moments of the former forces round the point q is—



$$P' \cdot qP' - \frac{qB}{AB}\, w \cdot \frac{qB}{2}$$

Now the reactions at P and P′ are

$$\frac{b'w}{b+b'}\,,\quad \frac{bw}{b+b'}$$

consequently the sum of the moments in question

$$= \frac{bw}{b+b'}\,(b'-x) - \frac{w}{2a}\left(\frac{a}{2}-x\right)^2$$

$$= \frac{w}{2}\left\{\frac{2bb'}{b+b'} - \frac{a}{4} + \frac{b'-b}{b'+b}\,x - \frac{x^2}{a}\right\} \qquad (5)$$

and this is equal to the sum of the moments of the elastic forces at the section q, consequently

$$\frac{ak^2}{6a}\cdot\frac{1}{\varrho} = \frac{2bb'}{b+b'} - \frac{a}{4} + \frac{b'-b}{b'+b}\,x - \frac{x^2}{a} \qquad (6)$$

The right hand member of this equation is equivalent to

$$-\frac{1}{a}\left(x - \frac{a}{2}\cdot\frac{b'-b}{b'+b}\right)^2 + 2bb'\,\frac{b+b'-\frac{a}{2}}{(b'+b)^2} \qquad (7)$$

In order, therefore, that there be any points in the bar of *no curvature*, we have

$$b + b' > \frac{a}{2}$$

that is unless the points of support are further apart than half the length of the bar, the whole bar will be convex upwards.

If one support be under one extremity of the bar and the other support at one third the bar's length from the other extremity, the bar will, for one half of its length, be convex upwards, and for the other half concave upwards; for if $b = \frac{a}{2}\ b' = \frac{1}{6}\,a$ the expression (7) becomes—

$$\left(x + \frac{a}{4}\right)^2 - \frac{a^2}{16}$$

which is manifestly negative from $x = 0$ to $x = -\frac{a}{2}$ and positive from $x = 0$ for any positive value of x. It is, however, to be remembered, that the equation (6) represents only that part of the bar which lies between the points of support PP'. For any point in $P'B$ the moment of the forces is simply

$$-\left(\frac{a}{2}-x\right)^2\frac{w}{2a}$$

consequently the equation of $P'B$ is

$$\frac{ak^2}{6\alpha}\cdot\frac{1}{\varrho}=-\frac{a}{4}+x-\frac{x^2}{a} \tag{8}$$

As the co-efficient of $\frac{1}{\varrho}$ in this equation is of frequent recurrence, we shall put

$$\frac{ak^2}{6\alpha}=\frac{1}{\mu} \tag{9}$$

Returning now to equation (6), and substituting for the radius of curvature its expression in x and y, we have

$$\frac{\frac{d^2y}{dx^2}}{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}=\mu\left(\frac{2bb'}{b+b'}-\frac{a}{4}+\frac{b'-b}{b'+b}x-\frac{x^2}{a}\right)$$

$$\therefore\frac{\frac{dy}{dx}}{\left(1+\frac{dy^2}{dx^2}\right)^{\frac{1}{2}}}=\mu\left(\left[\frac{2bb'}{b+b'}\quad\frac{a}{4}\right]x+\frac{b'-b}{b'+b}\,\frac{x^2}{2}\quad\frac{x^3}{3a}\right)+i \tag{10}$$

where i is the value of the quantity $\frac{dy}{dx}\left(1+\frac{dy^2}{dx^2}\right)^{-\frac{1}{2}}$ when $x=0$.

From (10) we can express $\frac{dy}{dx}$ as a function of x, but it will be impossible to integrate the equation. This, however, is of no practical consequence for $\frac{dy}{dx}$ is so small a quantity that its cube may well be neglected, and we may put

$$\frac{1}{\mu}\cdot\frac{dy}{dx}=\frac{i}{u}+\left(\frac{2bb'}{b+b'}-\frac{a}{4}\right)x+\frac{b'-b}{b'+b}\,\frac{x^2}{2}-\frac{x^3}{3a} \tag{11}$$

$$\therefore\frac{y}{\mu}=\frac{\delta}{u}+\frac{i}{u}x+\left(\frac{2bb'}{b+b'}-\frac{a}{4}\right)\frac{x^2}{2}+\frac{b'-b}{b'+b}\,\frac{x^3}{6}-\frac{x^4}{12a} \tag{12}$$

where δ is the value of y, when $x=0$.

If in this equation we make first $x=b'$, then $x=-b$, the corresponding values of y are each $=0$. Hence two equations from which δ and i may be expressed in known quantities; the results are

$$0=\frac{i}{\mu}+\frac{b'-b}{2}\left\{\frac{b^2+5bb'+b'^2}{3(b+b')}-\frac{b^3+b'^2}{6a}-\frac{a}{4}\right\} \tag{13}$$

$$0=\frac{\delta}{\mu}+\frac{bb'}{2}\left\{\frac{b^2+4bb'+b'^2}{3(b+b')}-\frac{b^2-bb'+b'^2}{6a}-\frac{a}{4}\right\} \tag{14}$$

Let p' be the value of $\frac{dy}{dx}$ at P′, then from (11) and (13) we obtain, eliminating i,

$$\frac{p'}{\mu} = \frac{b + b'}{2}\left\{\frac{-b^2 + 2bb' - 3b'^2}{6a} + \frac{b + 2b'}{3} - \frac{a}{4}\right\} \tag{15}$$

Integrating twice equation (8) we have—

$$\frac{1}{\mu}\frac{dy}{dx} = C - \frac{a}{4}x + \frac{x^2}{2} - \frac{x^3}{3a} \tag{16}$$

$$\frac{y}{\mu} = C' + Cx - \frac{a}{8}x^2 + \frac{x^3}{6} - \frac{x^4}{12a} \tag{17}$$

putting in these equations $x = b'$, the left hand members will become $\frac{p'}{\mu}$ and o; thus we have two equations for determining C and C′, and the values are

$$C = \frac{b' - b}{2}\left\{\frac{b^2 + b'^2}{6a} + \frac{a}{4}\right\} + \frac{b^2 + 3bb' - b'^2}{6} \tag{18}$$

$$C' = -\frac{bb'}{2}\left\{\frac{3b' + b}{3} - \frac{a}{4} - \frac{b^2 - bb' + b'^2}{6a}\right\} \tag{19}$$

And finally the equations of the two curves are

1st for PP′—

$$\left.\begin{aligned}\frac{y}{\mu} = &-\frac{bb'}{2}\left\{\frac{b^2 + 4bb' + b'^2}{3(b + b')} - \frac{b^2 - bb' + b'^2}{6a} - \frac{a}{4}\right\}\\ &-\frac{b' - b}{2}\cdot\left\{\frac{b^2 + 5bb' + b'^2}{3(b + b')} - \frac{b^2 + b'^2}{6a} - \frac{a}{4}\right\}x\\ &+\left\{\frac{bb'}{b + b'} - \frac{a}{8}\right\}x^2 + \frac{1}{6}\left\{\frac{b' - b}{b' + b}\right\}x^3 - \frac{1}{12a}x^4\end{aligned}\right\} \tag{20}$$

2d for P′B—

$$\left.\begin{aligned}\frac{y}{\mu} = &-\frac{bb'}{2}\left\{\frac{b + 3b'}{3} - \frac{b^2 - bb' + b'^2}{6a} - \frac{a}{4}\right\}\\ &+\frac{b' - b}{2}\cdot\left\{\frac{b^2 + 3bb' - b'^2}{3(b' - b)} + \frac{b^2 + b'^2}{6a} + \frac{a}{4}\right\}x - \frac{a}{8}x^2 + \frac{x^3}{6} - \frac{1}{12a}x^4\end{aligned}\right\} \tag{21}$$

It is almost unnecessary to remark that this last equation will, with the proper interchange of symbols, express the curve of AP.

3.

Let us now consider two particular cases; and first, when $b = b' = 0$: the case of bar supported at its centre. The equation is (21)

$$\frac{y}{\mu} = -\frac{a}{8}x^2 + \frac{1}{6}x^3 - \frac{1}{12a}x^4 \tag{22}$$

This curve is wholly convex upwards. In order to ascertain how much the projection of the neutral axis upon a horizontal plane is shorter than the neutral axis itself, we must put

$$s = 2\int_0^{\frac{a}{2}}\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}}dx$$

$$\therefore s - a = \int_0^{\frac{a}{2}}\frac{dy^2}{x^2}dx \tag{23}$$

Now

$$\frac{dy}{dx} = \mu \left(- \frac{a}{4} x + \frac{1}{2} x^2 - \frac{1}{3a} x^3\right) \tag{24}$$

$$\therefore s - a = \mu^2 \int^a \left(- \frac{a}{4} x + \frac{1}{2} x^2 - \frac{1}{3a} x^3\right)^2 dx$$

$$= \frac{\mu^2 a^5}{14 \cdot 128}$$

$$= \frac{36\, a^3 \alpha^2}{14 \cdot 128 k^4}$$

$$= \frac{9}{448} \cdot \frac{a^3 \alpha^2}{k^4} \tag{25}$$

To take a particular example: Suppose a bar of iron 40 inches in length, an inch square in section, and weighing 10 lbs. The force necessary to extend such a bar the millionth part of its length is about 18 or say 20 lbs.; therefore the extension due to $w = 10$ lbs. is,—

$$\alpha = \frac{40}{2000000} \text{ inch.}$$

Substituting this value, with $k = 1$, in (25) we get for the apparent shortening of the neutral axis

$$\frac{360}{448} \cdot \left(\frac{8}{10000}\right)^2 = \frac{18}{35} \left(\frac{\text{inch}}{1000000}\right)$$

or half a millionth of an inch: a quantity inappreciable in ordinary micrometer microscopes.

But if there be marks on the upper surface of the bar, one at each extremity, these marks will be separated by the extension of the upper surface to the amount $-kp$, where p is the value of $\frac{dy}{dx}$ at the extremities; now $k = 1$ and p by (24) is,—

$$p = -\frac{\mu a^2}{24}$$

Substituting the value of μ the extension becomes

$$\frac{a\alpha}{4} \tag{26}$$

which in the particular case under consideration is the one five-thousandth part of an inch: being six or seven micrometer divisions—a very sensible quantity.

Again, consider the case of a bar supported at its extremities. The equation is obtained from (20) by making $b = \frac{a}{2} = b'$

$$\frac{y}{\mu} = -\frac{5}{192} a^3 + \frac{a}{8} x^2 - \frac{1}{12a} x^4 \tag{27}$$

$$\frac{1}{\mu} \frac{dy}{dx} = \frac{a}{4} x - \frac{1}{3a} x^3$$

$$s = 2\int_0^{\frac{a}{2}} \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} dx = a + \int_0^{\frac{a}{2}} \frac{dy^2}{dx^2} dx$$

Consequently the difference between the length of the curve formed by the neutral axis and its horizontal projection is

$$\mu^2\int_0^{\frac{a}{2}}\left(\frac{ax}{4}-\frac{x^3}{3a}\right)^2 dx = \frac{17\,\mu^2 a^5}{32\cdot 315} \tag{28}$$

and substituting the value of μ this becomes

$$\frac{17\,a^3\,\alpha^2}{280} \tag{29}$$

The value of this on the previous supposition of $a = 40$, $k = 1$, is

$$\frac{272}{175}\left(\frac{\text{inch}}{1000000}\right)$$

something more than a millionth of an inch, a quantity not discernible in the ordinary micrometer microscopes.

In this position, being supported at the extremities, the upper surface of the bar is wholly concave, and in a state of compression; consequently points upon its upper surface, and at the extreme ends of the bar, will be caused to approach each other by the quantity p, where p is the value of $\frac{dy}{dx}$; when $x = \frac{a}{2}$, this equals

$$\mu\left\{\frac{a^2}{8}-\frac{a^2}{24}\right\} = \frac{\mu a^2}{12}$$

$$= \frac{\alpha a}{2} \tag{30}$$

Comparing this with (26) we see that if a bar be supported at its extremities the *contraction* of the upper surface between the extreme points is double the *extension* of the same upper surface when the bar is suspended from the centre. This result we shall see to be borne out by actual measure.

If the bar be supported, as is very common, on rollers at one fourth and three fourths of its length, the extension of the upper surface is

$$\frac{\alpha a}{16} \tag{31}$$

a sensible quantity.

It appears, then, that by altering the points of support a bar may measure considerably longer or shorter than it would if laid upon a horizontal surface, and also that the alteration of length does not in any degree proceed from the actual curving of the neutral axis, but from the direction of the tangents to this line at its extremities. If the bar be so supported that these tangents are parallel, there will be no alteration of the distance between points on the upper surface of the bar at its extremities. By differentiating (21) and putting $x = \frac{1}{2}a$, we get for the tangent of the direction of the neutral axis at B,

$$\mu\left\{\frac{b^2+3\,bb'-b'^2}{6}+\frac{b'-b}{2}\left(\frac{b^2+b'^2}{6a}+\frac{a}{4}\right)-\frac{a^2}{24}\right\} \tag{32}$$

and substituting b for b', and *vice versâ*, we get the tangent of the direction of the other extremity of the axis; viz.,

$$\mu\left\{\frac{b'^2+3\,bb'-b^2}{6}+\frac{b-b'}{2}\left(\frac{b^2+b'^2}{6a}+\frac{a}{4}\right)-\frac{a^2}{24}\right\} \tag{33}$$

The sum of these two is to be made zero; that is,

$$bb' - \frac{a^2}{12} = 0 \qquad (34)$$

If this relation hold good between the distance of the supports from the centre of the bar, the distance between points engraved on its upper surface at its extremities will be the same as if the bar be lying on a horizontal plane. If the supports be equidistant from the centre, and therefore $b = b'$,

$$b = \frac{a}{2\sqrt{3}} \qquad (35)$$

This formula expresses the proper distance of the supports from the centre of the bar, in order that there may be no difference between the apparent length of the bar when so supported, and when lying on a horizontal plane; in other words, the total extension of the upper surface is zero, and the direction of the neutral axis at either extremity is horizontal. This result is due to the Astronomer Royal, and is a particular case of the following more general theorem: If a uniform bar be supported on n equidistant rollers, exerting equal pressure upwards (as in the lever system), and if the distance apart of the rollers be

$$\frac{a}{\sqrt{n^2 - 1}}$$

the total extension of the upper surface between its extreme points is zero. (See *Memoirs of the Royal Astronomical Society, Vol.* XV.)

From (34) we learn further that if a bar supported on two rollers at the distance expressed in (35) be slightly displaced by rolling a small distance to the right or left, the effect will be immaterial.

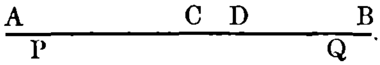
In general, from the sum of (32) and (33) multiplied by $\frac{1}{2}k$, we have the following: If a bar be supported at two points whose distances on either side of the centre are b, b', the total extension of the upper surface is

$$\frac{\alpha a}{k}\left\{\frac{1}{4} - 3\frac{bb'}{a^2}\right\} \qquad (36)$$

where a is the length, and k the depth of the bar, and α the small quantity by which the bar would be elongated or compressed by the direct action of a force equal to its own weight.

4.

Let us now consider the result of a pressure applied in a vertical direction to a given point in the upper surface of the bar. Let AB be the bar resting upon supports at P and Q equally distant from its extremities. Let the pressure act at D, the distance CD from the centre being b', and CQ = CP = b.



Let the pressure $= \theta w$, w being the weight of the bar. Let the axis of co-ordinates x pass through P and Q, the value of x corresponding to C being zero. We have first to obtain the pressures P and Q upon the supports from the conditions of equilibrium of the bar. These are given by the equations

$$P + Q = w + \theta w$$

$$-P + Q = \theta w \frac{b'}{b}$$

from which

$$P = \frac{1}{2}w + \frac{1}{2}\theta w\left(1 - \frac{b'}{b}\right) \qquad (37)$$

$$Q = \frac{1}{2}w + \frac{1}{2}\theta w\left(1 + \frac{b'}{b}\right) \qquad (38)$$

For any point in PD the sum of the moments of the bending forces is

$$-\theta w\,(b'-x)+Q\,(b-x)-\frac{w}{2a}\left(\frac{a}{2}-x\right)^2$$

$$=\frac{\theta w}{2}(b-b')+\frac{w}{2}\left(b-\frac{a}{4}\right)+\frac{\theta w}{2}\left(1-\frac{b'}{b}\right)x-\frac{w}{2a}x^2$$

and for any point in DQ the sum of the moments is

$$=\frac{\theta w}{2}(b+b')+\frac{w}{2}\left(b-\frac{a}{4}\right)-\frac{\theta w}{2}\left(1+\frac{b'}{b}\right)x-\frac{w}{2a}x^2$$

Hence the differential equations of the two curves PD and DQ are:

$$\frac{1}{\mu}\frac{d^2y}{dx^2}=\theta\,(b-b')+b-\frac{a}{4}+\theta\left(1-\frac{b'}{b}\right)x-\frac{x^2}{a} \qquad (39)$$

$$\frac{1}{\mu}\frac{d^2y}{dx^2}=\theta\,(b+b')+b-\frac{a}{4}-\theta\left(1-\frac{b'}{b}\right)x-\frac{x^2}{a} \qquad (40)$$

Integrating once we have—

$$\text{PD} \ldots\ \frac{1}{\mu}\frac{dy}{dx}=C+\left(b+\theta b-\theta b'-\frac{a}{4}\right)x+\frac{\theta}{2}\left(1-\frac{b'}{b}\right)x^2-\frac{x^3}{3a} \qquad (41)$$

$$\text{DQ} \ldots\ \frac{1}{\mu}\frac{dy}{dx}=C'+\left(b+\theta b+\theta b'-\frac{a}{4}\right)x-\frac{\theta}{2}\left(1+\frac{b'}{b}\right)x^2-\frac{x^3}{3a} \qquad (42)$$

The difference of these equations being zero for $x=b'$, we get by substituting this value of x,

$$C-C'=\theta b'^2 \qquad (43)$$

and integrating a second time—

$$\text{PD} \ldots\ \frac{y}{\mu}=C_{,}+Cx+\tfrac{1}{2}\left(b+\theta b-\theta b'-\frac{a}{4}\right)x^2+\frac{\theta}{6}\left(1-\frac{b'}{b}\right)x^3-\frac{x^4}{12a} \qquad (44)$$

$$\text{DQ} \ldots\ \frac{y}{\mu}=C'_{,}+C'x+\tfrac{1}{2}\left(b+\theta b-\theta b'-\frac{a}{4}\right)x^2-\frac{\theta}{6}\left(1-\frac{b'}{b}\right)x^3-\frac{x^4}{12a} \qquad (45)$$

the difference of these equations is zero for $x=b'$, and hence

$$C_{,}-C'_{,}=-\tfrac{1}{3}\theta b'^3 \qquad (46)$$

Also from (44), (45), making $x=-b$ and $x=b$, we have two equations which, taken with (43) and (46), determine the values of $C_{,}$ $C'_{,}$ C C'. They are as follows:—

$$C=\theta b'b'\left\{-\tfrac{1}{3}+\tfrac{1}{2}\frac{b'}{b}-\tfrac{1}{6}\frac{b'^2}{b^2}\right\} \qquad (47)$$

$$C'=\theta bb'\left\{-\tfrac{1}{3}-\tfrac{1}{2}\cdot\frac{b'}{b}-\tfrac{1}{6}\cdot\frac{b'^2}{b^2}\right\}$$

$$C_{,}=\frac{b^2}{2}\left\{\frac{a}{4}-b\,(1+\tfrac{2}{3}\theta)+\frac{b^2}{6a}\right\}+\frac{b'^2}{2}\theta\,(b-\tfrac{1}{3}b')$$

$$C'_{,}=\frac{b^2}{2}\left\{\frac{a}{4}-b\,(1+\tfrac{2}{3}\theta)+\frac{b^2}{6a}\right\}+\frac{b'^2}{2}\theta\,(b+\tfrac{1}{3}b')$$

Let q and p represent the values of $\frac{dy}{dx}$ at the points of support Q and P respectively, then if in (41) and (42) we put $x = -b$ and $x = b$ respectively, and take the difference, we get

$$\frac{q-p}{\mu} = \theta\,(b^2 - b'^2) - 2 \cdot \frac{b}{a}\left(\frac{a^2}{4} - ab + \frac{b^2}{3}\right) \tag{48}$$

consequently the compression of the upper surface between P and Q is

$$\frac{3\,\alpha\theta}{ak}\,(b^2 - b'^2) - \frac{6\,\alpha b}{a^2 k}\left(\frac{a^2}{4} - ab + \frac{b^2}{3}\right) \tag{49}$$

the effect of the pressure alone being

$$\frac{3\,\alpha\theta}{ak}\,(b^2 - b'^2) \tag{50}$$

Let us next ascertain the effect of the pressure in shortening (through its curvature) the horizontal projection of the neutral axis of the bar. Let us suppose the pressure to be applied at the centre so that $b' = 0$, and suppose the bar supported at its extremities so that $b = \frac{1}{2}\,a$. In this case one equation will express the whole curve of the bar: this equation is—

$$\frac{y}{\mu} = -\frac{a^3}{8}\left(\tfrac{5}{24} + \frac{\theta}{3}\right) + a\left(\tfrac{1}{8} + \frac{\theta}{4}\right)x^2 - \frac{\theta}{6}\,x^3 - \frac{x^4}{12\,a} \tag{51}$$

$$\text{or,}\quad \frac{1}{\mu}\,\frac{dy}{dx} = a\left(\tfrac{1}{4} + \frac{\theta}{2}\right)x - \frac{\theta}{2}\,x^2 - \frac{1}{3\,a}\,x^3$$

$$\frac{dy^2}{dx^2} = \frac{\alpha^2}{k^4}\left\{\left(\tfrac{3}{2} + 3\,\theta\right)x - \frac{3\,\theta}{a}\,x^2 - \frac{2}{a^2}\,x^3\right\}^2$$

and the difference of length between the vertical axis and its projection will be—

$$\frac{\alpha^2}{k^4}\int_0^{\frac{a}{2}}\left\{\left(\frac{3}{2} + 3\,\theta\right)x - \frac{3\,\theta}{a}\,x^2 - \frac{2\,x^3}{a^2}\right\}^2 dx$$

$$= \frac{\alpha^2\,a^3}{8\,k^4}\left\{\frac{34}{70} + \frac{61}{40}\,\theta + \frac{6}{5}\,\theta^2\right\} \tag{52}$$

To take a particular case: suppose, as before, $a = 40$, $k = 1$,

$$\alpha = \frac{20}{1000000}$$

the weight of the bar being 10 lbs., and let the pressure applied at the centre be also 10 lbs., so that $\theta = 1$. Then by (52) the shortening of the bar through direct curvature

$$= \frac{899}{7} \cdot \frac{8}{100000000} = \frac{\text{inch}}{100000}$$

which is about a third of a division in ordinary micrometer microscopes,—a quantity only just visible.

But the contraction of the upper surface resulting from the pressure at the centre is by (50)

$$\frac{3\,\alpha a}{4} = \frac{6\ \text{inch}}{10000}$$

which is about 20 divisions.

5.

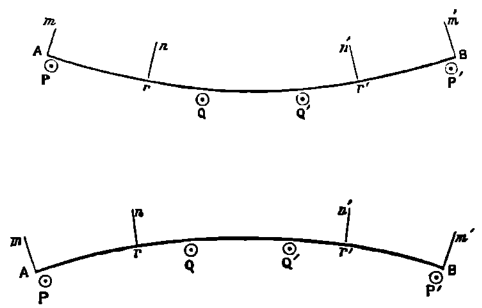
In order to ascertain to what extent the results obtained in the preceding sections agree with observations, experiments were made on three bars of iron, of which the following are the descriptions :—

1. Cast-iron bar marked Q; the faces severally marked I, II, III, IIII. The section is a rectangle; the breadth of the faces I, and III, is 0·988 inch, and that of the faces II, IIII, is 0·986 inch, the length is 40·30 inches. The faces were carefully planed and polished. The bar was cast at an inclination of 45° to the horizon; the casting is not absolutely perfect, there being a small hole in one of the faces. The weight of this bar is 10·16 lbs.

2. Cast-iron bar marked R; the faces severally marked I, II, III, IIII, the section is a square; the breadth of each face is 0·990 inch, and the length is 40·30 inches. The faces are carefully planed and polished. The bar was cast in a horizontal position, and is, though a good casting, less perfect than Q. The weight is 10·19 lbs.

3. Wrought Swedish iron bar marked S. The faces carefully planed and polished are marked I, II, III, IIII, the section is a rectangle; the breadth of faces I, and III, is 0·992 inch, and that of faces II, IIII, is 0·957 inch; the length is 40·30 inches and the weight 10·87 lbs. The material is of very excellent quality, and has the appearance of steel.

The method of conducting the experiment was this—



AB is an elastic bar having under it four supports PQQ′P′ capable of motion in a vertical direction only. In the upper figure the outer supports PP′ are raised and the inner supports QQ′ withdrawn from below, so that the bar rests only on PP′. If the outer supports be now lowered and the inner raised we have the state represented in the lower figure, where the outer supports are withdrawn, and the bar is carried by the inner supports only. A*m* B*m′* *rn* *r′n′* are perpendiculars raised from the surface of the bar and rigidly connected with it, so that when the bar is bent the perpendiculars remain constantly at right angles to the bar at their base. These perpendiculars are of wire secured to the surface of the bar, and on the top of each a fine dot is engraved. It will be seen that the distance of these dots at *m n n′ m′* will vary considerably more than corresponding points on the surface of the bar. If p be the length of the perpendiculars, k the depth of the bar, then the effects of flexure in AB are increased in the proportion of $p + \frac{1}{2} k : \frac{1}{2} k$. Over the points *mn n′m′* are adjusted micrometer microscopes, and by these are actually measured the changes of length of *mm′ mn′ nm′* as the supports are altered. The supports QQ′ are equidistant from the centre, and the bar may rest in four different ways, viz., on

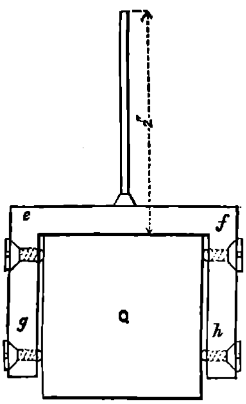
PP′, PQ′, QP′, QQ′.

It is essential to the obtaining of trustworthy results, that the transfer of the bar from one pair of rollers to another be easily accomplished, that is, without the bar being handled; this will be effected if we are able to raise or lower any of the rollers without touching the bar.

Each of the rollers actually employed is a cylinder of brass, an inch long and an inch in diameter, having a small flange to prevent the bar slipping off, not however *exactly* a cylinder, as two of them are slightly concave, and the other two slightly convex or barrel shaped, the object of this being that the bar may always get a true bearing or be supported on three points. The rollers are mounted each in a small brass framework, in such a manner that by the turning of a vertical screw *nnf* (see Figure 7, Plate VI.) the roller *k* is drawn upwards or let down. The screw is of fine thread, and great pains were taken in the construction to make the roller firm and free from all shake, while its motion by means of the milled-headed screw is easy.

The roller frames are screwed down in any required position in the bottom of a box specially made for these experiments. The length of the box is four feet; along each of the inner sides is marked a scale of inches and parts, so that the rollers may be placed with great accuracy at any required distances from the centre of the bar. The two essential points of the adjustment of the rollers are that they be all four truly in line, and that the axis of each be truly perpendicular to that line. They are then firmly screwed down to brass plates in the bottom of the box. The latter rests on the camels of the comparison apparatus, and so can be adjusted laterally or longitudinally under the microscope.

Four microscopes were arranged in line over the bar; the distance of the outer microscopes H, K was exactly 40 inches from centre to centre. The inner microscopes A, C were 20 inches apart from centre to centre, each being midway between the centre of the bar, and one of its extremities The microscopes were carefully aligned, and their axes made truly vertical.



The perpendiculars to the surface of the bar were affixed as in the annexed figure, where Q represents the bar in end elevation, *efgh* a piece of brass nearly fitting three sides of the bar, but allowing a little transverse play. It can be fixed and held tight by the four small screws when adjusted into line. To the upper surface of this brass is rigidly fixed in a vertical position a piece of iron wire, and on the top of this wire a piece of platinum is let in. On the platinum is engraved a fine dot. Four of these perpendiculars are attached to the bar at distances of 10 and 20 inches precisely on either side of the centre; thus the outer ones are each 0·15 of an inch from the extremity of the bar, the length of which is 40·30 inches.

The supporting rollers were placed at various times in the following positions:—

1. At 20 inches left and right of the centre; the supports so placed are designated EE′ respectively.
2. At 2 inches left and right of the centre; designated CC′.
3. At 11·63 inches left and right of the centre; designated NN′, this being the normal position for two supports computed from the formula (35).
4. At 6·72 inches left and right of the centre; designated SS′, being one sixth the length of the bar distant from the centre.

The bar was not observed in these positions only, but on unsymmetrically placed supports. For instance, suppose the four rollers are fixed in the positions ESS′E′ (the accented letters invariably denote supports to the *right* of the centre), then the bar is compared with itself in four different positions:—

1st on	. . . . .	EE′
2d „	. . . . .	SS′
3d „	. . . . .	ES′
4th „	. . . . .	SE′

and it has been shown that in these last two positions the bar is for one half concave, and the other half convex.

6.

On referring to the figures, page 31, it will be seen that the displacement of the top of a perpendicular is—

$$(p + \tfrac{1}{2}k)\,\frac{dy}{dx}$$

where p is the length of the perpendicular and $\frac{dy}{dx}$ the inclination of the bar at the base of the perpendicular. The actual length of the perpendiculars employed is two inches precisely, hence the displacement is—

$$(2 + \tfrac{1}{2}k)\,\frac{dy}{dx}$$

If we differentiate the equations 20 and 21, substituting the proper value of a, b, b', we get for that portion of the bar which includes its centre, the following values of $\frac{1}{\mu}\frac{dy}{dx}$ for 10 different dispositions of the supports.

Supports.	Values of $\frac{1}{\mu}\cdot\frac{dy}{dx}$
CC′	$- 8{\cdot}0750\,x - {\cdot}008272\,x^3$
NN′	$+ 1{\cdot}5550\,x - {\cdot}008272\,x^3$
EE′	$+ 9{\cdot}9250\,x - {\cdot}008272\,x^3$
NC′	$- 21{\cdot}19 - 6{\cdot}6620\,x - {\cdot}35325\,x^2 - {\cdot}008272\,x^3$
CN′	$+ 21{\cdot}19 - 6{\cdot}6620\,x + {\cdot}35325\,x^2 - {\cdot}008272\,x^3$
EN′	$+ 23{\cdot}47 + 4{\cdot}6326\,x - {\cdot}13230\,x^2 - {\cdot}008272\,x^3$
NE′	$- 23{\cdot}47 + 4{\cdot}6326\,x + {\cdot}13230\,x^2 - {\cdot}008272\,x^3$
SS′	$- 3{\cdot}3584\,x - {\cdot}008272\,x^3$
ES′	$+ 13{\cdot}40 - 0{\cdot}0188\,x - {\cdot}24861\,x^2 - {\cdot}008272\,x^3$
SE′	$- 13{\cdot}40 - 0{\cdot}0188\,x + {\cdot}24861\,x^2 - {\cdot}008272\,x^3$

and for the portion of the bar to the right of the right support, or to the left of the left support (after the proper change of sign):

CC′	$+ 2{\cdot}00 - 10{\cdot}0750\,x + {\cdot}50000\,x^2 - {\cdot}008272\,x^3$
NN′	$+ 67{\cdot}67 - 10{\cdot}0750\,x + {\cdot}50000\,x^2 - {\cdot}008272\,x^3$
EE′	$+ 200{\cdot}00 - 10{\cdot}0750\,x + {\cdot}50000\,x^2 - {\cdot}008272\,x^3$
NC′	$- 17{\cdot}79 - 10{\cdot}0750\,x + {\cdot}50000\,x^2 - {\cdot}008272\,x^3$
CN′	$+ 41{\cdot}04 - 10{\cdot}0750\,x + {\cdot}50000\,x^2 - {\cdot}008272\,x^3$
EN′	$+ 109{\cdot}00 - 10{\cdot}0750\,x + {\cdot}50000\,x^2 - {\cdot}008272\,x^3$
NE′	$+ 123{\cdot}60 - 10{\cdot}0750\,x + {\cdot}50000\,x^2 - {\cdot}008272\,x^3$
SS′	$+ 22{\cdot}56 - 10{\cdot}0750\,x + .50000\,x^2 - {\cdot}008272\,x^3$
ES′	$+ 47{\cdot}17 - 10{\cdot}0750\,x + {\cdot}50000\,x^2 - {\cdot}008272\,x^3$
SE′	$+ 87{\cdot}16 - 10{\cdot}0750\,x + {\cdot}50000\,x^2 - {\cdot}008272\,x^3$

The quantities obtained from this table by making $x \pm 10$ and ± 20, will, when multiplied by $\mu\,(2 + \tfrac{1}{2}k)$ or (equation 9) by—

$$\frac{3\alpha}{ak^2}\,(4 + k) \qquad (53)$$

give the disturbances of the tops of the perpendiculars.

The numerical quantities to be multiplied by (53) are found to be—

Supports.	m	n	n'	m'
CC′	− 65·68	− 57·02	− 57·02	− 65·68
NN′	− 0·01	+ 7·28	+ 7·28	− 0·01
EE′	+ 132·32	+ 90·98	+ 90·98	+ 132·32
NC′	− 26·64	− 18·38	− 76·81	− 85·47
CN′	− 85·47	− 76·81	− 18·38	− 26·64
EN′	+ 55·93	+ 27·82	+ 48·30	+ 41·32
NE′	+ 41·32	+ 48·30	+ 27·82	+ 55·93
SS′	− 45·12	− 36·46	− 36·46	− 45·12
ES′	+ 19·48	+ 3·00	− 11·85	− 20·51
SE′	− 20·51	− 11·85	+ 3·00	+ 19·48

The disturbances are measured, positively, inwards towards the centre of the bar. The columns m, n, n', m', refer to the perpendiculars so marked, in order from left to right in the figure, page 31.

If $[m, m']$ signify the diminution of distance of the tops of the two outer perpendiculars when the bar is supported on two given supports, as compared with their distance when the bar lies undisturbed on a horizontal plane, and if $[m\ n']$ $[n\ m']$, signify the same for the distances $m\ n'$, $n\ m'$, then their actual values will be as follows :—

Supports.	$[m\ n']$	$[m\ m']$	$[n\ m']$
CC′	− 122·70	− 131·36	− 122·70
NN′	+ 7·27	− 0·02	+ 7·27
EE′	+ 223·30	+ 264·64	+ 223·30
NC′	− 103·45	− 112·11	− 103·85
CN′	− 103·85	− 112·11	− 103·45
EN′	+ 104·23	+ 97·25	+ 69·14
NE′	+ 69·14	+ 97·25	+ 104·23
SS′	− 81·58	− 90·24	− 81·58
ES′	+ 7·63	− 1·03	− 17·51
SE′	− 17·51	− 1·03	+ 7·63

(54)

where each quantity is to be multiplied by—

$$\frac{3\alpha\,(4 + k)}{ak^2}$$

7.

The first observations were made on Q. The section of this bar may be taken as a square, the difference of the faces being very minute; also $k = 0·987$.

The method of procedure is as follows :—the bar resting on NN′ is made level by the elevation or depression of the supporting rollers NN′; the microscopes are then adjusted over the dots (which have themselves been carefully aligned), their axes made vertical, and the micrometer screw adjusted carefully parallel to the length of the bar. In each position of the bar the four microscopes are read twice, viz., in the order H, A, C, K, K C, A, H. The values of one division of the micrometer in these microscopes are, in *millionths of a yard*,

for H . . . . 0·7949
„ A . . . . 1·177
„ C . . . . 0·869
„ K . . . . 0·7980

The observations in a single visit to the bar-room are recorded as follows:—

Rollers.	H m	A n	C n'	K m'
	d	d	d	d
EE′	76·6	35·4	59·6	95·5
	76·8	36·3	60·5	96·4
CC′	20·7	6·6	3·3	19·6
	19·9	6·4	3·5	19·3
EE′	86·4	43·6	52·2	86·5
	86·5	43·4	52·0	87·4
CC′	21·5	5·5	3·5	19·5
	20·7	5·3	3·4	19·4

The microscopes being once adjusted to focus are not again altered, but in every case the dots are brought by the elevation or depression of the supporting rollers into the focus of the microscope. It is evident there must be always a slight error in focus to be dispersed among the dots on account of the flexure of the bar, the foci of the microscopes remaining in a fixed and horizontal straight line. The flexure of the bar, however, is not of sufficient magnitude to cause much trouble to the observer, though it is quite perceptible.

The following results are the means of four comparisons one on each face, expressed in *millionths of a yard.* $\Delta\,[mm']$ means the variation in the distance of the dots m, m', corresponding to the change of supports specified in the left-hand column of the table.

Change of Supports		$\Delta\,[mn']$	$\Delta\,[mm']$	$\Delta\,[nm']$
from	to			
NN′	EE′	57·8	70·7	60·7
„	EN′	25·0	25·9	17·4
„	NE′	16·4	26·8	28·5
„	CC′	− 36·2	− 35·8	− 35·7
„	NC′	− 31·8	− 31·8	− 32·2
„	CN′	− 31·7	− 31·8	− 32·4

The following sets of comparisons are with the face marked IIII uppermost.

Set.	Change of Supports from	to	Δ [mn′]	Δ [mm′]	Δ [nm′]	No. of Comparisons.
1	CC′	EE′	93·8	107·0	97·4	8
2	”	”	92·5	104·9	98·3	10
3	”	”	93·2	107·3	98·2	8
4	”	”	93·9	106·7	94·1	10
5	”	”	93·2	106·4	92·8	12

On the completion of the first set in this table it was observed that unsymmetrical results were being obtained, inasmuch as we should expect to find Δ [*mn′*] = Δ [*nm′*]. This might be explained by supposing one end of the bar less flexible than the other; consequently further series of comparisons were made with the view of ascertaining, by the reversal of the bar, the real cause of the discrepancy. In the second set, the end of the bar marked Q was, as in the first set, to the left. In the third set, the end marked Q was reversed, and placed to the *right*. The same peculiarity in the result still remains, as though the *left* end of the bar were less flexible; we have for Δ [*mn′*] 93·2 and for Δ [*nm′*] 98·2. It is therefore clear that the discrepancy is not in the iron, as we should have found it reversed on reversing the bar, which is not the case. Nor can the cause be sought in mere errors of observation; the testimony of the individual comparisons in each of the three first sets is uniform. Nor is it any peculiarity of an observer, as three different observers made the comparisons in regular routine. It is difficult to account for any unsymmetrical result, as there is nothing unsymmetrical in the apparatus or mode of observing, with this exception, that the rollers under the left half of the bar are concave, while those on the right are convex; so that on the left the bar is supported towards its outer edges, while on the right the contact of the supporting roller is at the centre of the breadth of the bar. The friction of the rollers is, through special attention in their construction, the least possible. The force which will just move the bar on its rollers is between four and six ounces.

Previous to the commencement of the set marked 4, a slight inclination in the direction of the micrometer screw of the right hand microscope to the true line of measurement was detected and adjusted.

In the fourth set, the mark Q was to the right, and in the fifth, Q was to the left. In these two sets the results are perfectly satisfactory. But the disappearance of the anomalous results is certainly not to be explained by the adjustment just referred to, as that cause is quite insufficient; moreover, it is only a temporary disappearance, as we shall see.

We shall reject the sets marked 1, 2, 3, in this table, and retain as the final result, the mean of the last two; viz.:—

Change of Supports.	Δ [mn′]	Δ [mm′]	Δ [nm′]	Number.
CC′ to EE′	93·5	106·5	93·4	22 comparisons.

If we now compare these observed changes of length with those obtained from the table (54), we can by the method of least squares obtain the value of the multiplier; the result is

$$\frac{3\alpha(4+k)}{ak^2} = 0\cdot2714$$

$$\therefore \alpha = 0\cdot7121$$

where α is the extension or compression of the bar Q under a force equivalent to its own weight (10·16 lbs.), and expressed in millionths of a yard.

This being the measure of elasticity of the bar, the observed and computed variations of length will stand as in the following table :—

Changes of Supports		Δ [mn′]		Δ [mm′]		Δ [nm′]	
from	to	Observed.	Computed.	Observed.	Computed.	Observed.	Computed.
NN′	EE′	57·8	58·6	70·7	71·8	60·7	58·6
"	EN′	25·0	26·3	25·9	26·4	17·4	16·8
"	NE′	16·4	16·8	26·8	26·4	28·5	26·3
"	CC′	− 36·2	− 35·3	− 35·8	− 35·7	− 35·7	− 35·3
"	NC′	− 31·8	− 30·0	− 31·8	− 30·4	− 32·2	− 30·1
"	CN′	− 31·7	− 30·1	− 31·8	− 30·4	− 32·4	− 30·4
CC′	EE′	93·5	93·9	106·5	107·5	93·4	93·9

8.

In the bar R, cast in a horizontal position, $k = 0\cdot990$ inch; the section being very accurately a square.

Two series of 8 and 10 comparisons respectively give for the change from CC′ to EE′ (face I uppermost.)

Δ [mn′]	Δ [mm′]	Δ [nm′]
91·2	105·4	95·6
92·7	104·8	96·0

Here we have again, with a different bar, the anomaly of Δ [mn′] < Δ [nm′]. The observations themselves are satisfactory enough. In a third series of 14 comparisons on the same face, the anomaly almost disappears. We have,—

Δ [mn′]	Δ [mm′]	Δ [nm′]
92·6	104·5	93·3

We shall take this to be the true representation of the effect of flexure.

With face I uppermost the bar was compared in its different positions on E, S, S′, E′, seven comparisons; and with the opposite face III uppermost, five comparisons on the same rollers. On comparing the observed and theoretical changes of length, we get for the multiplier,—

$$\frac{3\alpha(4+k)}{uk^2} = \cdot 2652$$

$$\alpha = \cdot 6997$$

This being the measure of elasticity of R, the observed and computed changes of length will stand as in the following table:—

Face of Bar.	Changes of Supports		Δ [mn′]		Δ [mm′]		Δ [nm′]	
	from	to	Observed.	Computed.	Observed.	Computed.	Observed.	Computed.
I	EE′	SS′	− 82·3	− 80·9	− 94·5	− 94·1	− 79·4	− 80·9
„	„	ES′	− 58·8	− 57·2	− 71·4	− 70·4	− 62·8	− 63·9
„	„	SE′	− 63·5	− 63·9	− 70·0	− 70·4	− 55·2	− 57·2
III	EE′	SS′	− 82·5	− 80·9	− 95·0	− 94·1	− 80·7	− 80·9
„	„	ES′	− 59·3	− 57·2	− 71·8	− 70·4	− 64·3	− 63·9
„	„	SE′	− 65·2	− 63·9	− 71·8	− 70·4	− 58·1	− 57·2
I	CC	EE′	+ 92·6	+ 91·8	+ 104·5	+ 105·0	+ 93·3	+ 91·8

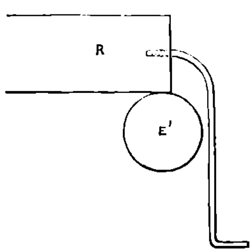
In order more satisfactorily to test the hypothesis of the equality of expansion and contraction of the material under the same directly applied force, or in other words to test the centrality of the neutral axis, the following experiment was made:—The bar resting on the extreme supports EE′, a weight of 9·45 lbs. was placed on the centre of the bar (covering just one square inch of the bar's surface), and the resulting variation in length repeatedly observed. With the face II. uppermost, the contraction of the distance between the tops of the outer perpendiculars,—

$$\Delta\,[mm'] = 96\cdot 34$$

with the opposite face IIII uppermost, there resulted,—

$$\Delta\,[mm'] = 96\cdot 58$$

Observations were then made upon a point two inches *below* the *under* surface of the bar to ascertain the extension of the under part of the bar, when weighted at the centre.



This was effected as follows:—Into the end of the bar was screwed firmly a piece of wire, bent into the form shown in the annexed figure, where R is the extremity of the bar, and E′ the right supporting roller. The lower part of the wire being bent in a horizontal direction, carried a small disk of platinum with an engraved dot at precisely 2 inches below the under surface of the bar. The microscopes being adjusted over these points, one at each extremity of the bar, the observations were proceeded with as before. The increase of length produced by the weight being placed on the centre of the bar (face I uppermost) is, from 5 comparisons,

$$\Delta\,[mm'] = 96\cdot 00$$

where as the contraction for the points above the surface we have found from the mean of the observations on faces II and III to be,—

$$\Delta\ [mm'] = 96 \cdot 46$$

This satisfactorily establishes the position of the neutral axis as at the centre of the bar.

The observed variations of the distance of the upper dots mm' when the weight is placed on different points of the bar's length is shown in the following table:—

On extreme Supports E E′

Face of Bar uppermost.	Position of weight.				
	Centre.	5 in. left of centre.	5 in. right of centre.	10 in. left of centre.	10 in. right of centre.
II IIII	96·34 96·58	90·7 90·8	90·6 90·8	71·7 73·0	72·9 72·4

Now from (48) it appears that if a bar be resting on supports at the distances $\pm b$ from the centre, and if a pressure $= \theta \times$ (weight of the bar) be applied at a distance $\pm b'$ from the centre, the effect upon points elevated to the height p above the surface of the bar at the distances $\pm b$ from the centre is,—

$$\mu\, \theta\, (p + \tfrac{1}{2}\, k)\, (b^2 - b'^2)$$

or,—

$$\frac{3\, \alpha\, \theta\, (4 + k)}{ak^2}\, (b^2 - b'^2) \qquad (55)$$

For the bar R this formula gives, for $b' = 0$, $b' = 5$, $b' = 10$, the following,—

$$98 \cdot 39;\quad 92 \cdot 24;\quad 73 \cdot 78$$

which are somewhat larger than the observed quantities. It must be borne in mind that this bar is not a very perfect casting.

9.

For faces I or III uppermost in the bar S, $k = \cdot 957$. With face I uppermost, and over the supports NCC′N′, seven comparisons were made; and with the face III uppermost, and over the supports ENN′E′, six comparisons were made; and finally with face III uppermost and over the supports ECC′E′ 16 comparisons were made. The observed variations of length being compared with the theoretical variations from table (54) we get,—

$$\frac{3\, \alpha\, (4 + k)}{ak^2} = \cdot 1722$$

$$\alpha = \cdot 4274$$

In the following table are collected the observed and computed variations of length.

Changes of Supports		Δ [mn']		Δ [mm']		Δ [nm']	
from	to	Observed.	Computed.	Observed.	Computed.	Observed.	Computed.
NN'	CC'	− 21·8	− 22·4	− 22·3	− 22·6	− 23·5	− 22·4
"	NC'	− 18·1	− 19·1	− 19·3	− 19·3	− 20·7	− 19·1
"	CN'	− 19·1	− 19·1	− 19·8	− 19·3	− 20·9	− 19·1
"	EE'	39·2	37·2	45·7	45·6	37·0	37·2
"	EN'	18·0	16·7	16·0	16·7	9·6	10·7
"	NE'	10·9	10·7	16·2	16·7	15·8	16·7
CC'	EE'	60·8	59·6	67·1	68·2	59·5	59·6

During the observations of which the results are given in the four last lines of this table, the rollers were reversed, that is, the concave-surfaced rollers which had been hitherto under the left end of the bar were placed on the right, and the convex on the left.

In the next series with this bar, three points only were observed; viz., *m m'* at the extremes; and a third, elevated above the precise centre of the bar,—we shall call this point *i*. It is at the same height above the surface as the others, namely, 2 inches exactly. The variations in length Δ [*mi*] and Δ [*im'*] of the two halves of the bar as the supports changed from EN' to NE' were then observed; ten comparisons with face II uppermost, and 12 comparisons with face IIII uppermost.

Face.	Change of Supports		Δ [*mi*]	Δ [*im'*]
	from	to		
II	EN'	NE'	− 9·7	9·8
IIII	"	"	− 9·7	9·9

The theoretical value is

$$\frac{3\alpha(4+k)}{\alpha k^2} \times 61{\cdot}55$$

Here $k = {\cdot}992$, which gives

$$-\Delta\,[mi] = 9{\cdot}9 = \Delta\,[im']$$

The effect of a weight placed at the centre while the bar rested on EE' was also observed, and the results are shown in the following table:—

Face.	Points.	Δ [mm']	
		Observed.	Computed.
I	Upper -	58·38	59·89
III	Upper -	58·27	59·89
I	Lower -	59·01	59·89

Of these results, the last is the mean of six observations, the former of two each. It is sufficiently clear that the neutral axis is exceedingly close to the centre of the bar.

10.

The differences which remain between the observed and computed variations of length in each of these bars, though really very minute quantities, are yet far too large to be attributed to mere errors of observation. They may be due, perhaps, in small part to the friction of the rollers, which is not easily subjected to calculation, as it cannot always be determined how it is acting. A more probable source of error is the tendency of the perpendiculars to ride upwards when being fastened on to the bar. This will be easily understood from the figure page 32. By the pressure of either of the lower screws, if not carefully applied, the little brass framework separates from contact with the upper surface of the bar. But this will not account for the principal differences. The runs of the microscopes are very well determined, and the inner microscopes A, C, were interchanged during the observations. The inevitable imperfections in focussing may have been a source of more error than the observers could have expected.

The total number of micrometer readings in these experiments is 2984.

IV.

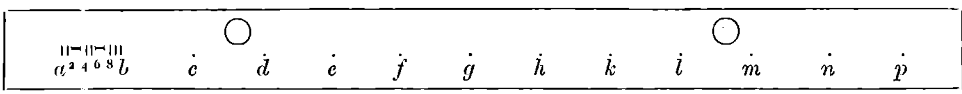
DETERMINATION OF THE

ERRORS OF DIVISIONS OF

OF.

1.

This scale, of which the upper surface is represented in the accompanying diagram, is divided into 12 inches; the thirteen lines are marked *a b c d e f g h k l m n p*. The extreme inches *ab* and *np* are divided each into tenths, and two of these tenths in each inch are again subdivided into hundredths.



The tenths in *ab* are numbered from *a* towards *b*. The space between the second and third lines is subdivided into hundredths, as is also the space between the sixth and seventh, as indicated by the dark lines. The inch *np* is similarly subdivided, but this part of the scale is not considered in the present investigation.

The lines of which the errors have been determined are

2 3 4 6 7 8 *b c d e f g*

together with the 18 lines subdividing into hundredths the two tenths.

The bar is supported on the same cast-iron stand on which the standard yard at other times lies. In Figure 5, Plate VI., the foot *f* is seen supported on the stand *bb′*. It rests immediately on a piece of iron *e e e* an inch broad and about a quarter of an inch thick, the middle part on which the bar rests being lower than the extremities. The extremities are supported on the two rollers which are held by the brass uprights *c c′*. The peculiar shape of the piece *e e e* is owing to the circumstance that the divisions on the standard yard are half an inch *below* its surface, while those of the foot are *on* its surface, and the vertical play of the screws which raise or lower the rollers is not sufficient to meet the requirements of the two bars.

The method of procedure is as follows:—The microscopes being set up at six inches apart, the spaces *ag*, *gp* are compared; this gives the error of the position of *g*. The microscopes are then set up at five inches apart, and the spaces *af*, *bg* compared. They are then set up at four inches apart, and the spaces *ae*, *bf*, *cg* compared; and, finally, the microscopes being at three inches apart, the spaces *ad*, *be*, *cf*, *dg* are compared.

The errors of the tenths of inches in *ab* were obtained as follows:—A small and beautifully divided silver scale, containing three tenths of an inch, was mounted to the left of the bar **OF**, and precisely in the same line with it, at the distance of about twelve inches from *ab*. The four lines on the silver scale marking three tenths of an inch are numbered 0 1 2 3, the tenths are subdivided into hundredths, which, however, we make no use of here. The silver scale and **OF** are both mounted on the block of wood *a a'* shown in Plate VI, Figure 5. This figure represents the mode of measurement of the small space of $\frac{1}{100}$ inch on the Contact Apparatus *h*, on the right of *f*, under the microscope K; but it will serve equally to explain the measurement of the tenths of inches on *a b*; for the small silver scale, mounted on a brass stand, occupied the place of *h* in Figure 5. The only difference is that the silver scale was on the left under H and the foot to the right under K. The small silver scale is secured to its brass stand (which is purposely weighty) by a couple of small springs pressing upon the extremities of its upper surface. It is also to be observed that **OF** can be moved some three quarters of an inch either to the right or left of its central position by causing the piece *e e e* to run upon the rollers which support its extremities.

First, suppose the lines 0 in the silver scale and *a* on **OF** to be brought, by the longitudinal running of the carriages *gg*, *g'g'*, into the centres of the fields of view of the microscopes, the former under H on the left, the latter under K on the right, and bisected by the cross-hairs of the micrometers, near the zeros. Then suppose the carriages to be run just two tenths of an inch to the left, we shall have the lines 2 on the silver scale and 2 on **OF** near the zeros of the microscopes; these lines being bisected, it is evident that we know the difference of length of the space 0 to 2 on the silver scale and *a* to 2 on **OF**. And in this manner we may compare any of the tenths on *ab* with the tenths on the silver scale, and so, by properly arranging the observations, obtain the relative lengths of the tenths of *ab*.

The lines subdividing *ab* with which we are most concerned are 2, 3, 6, 7, but we cannot determine these without at least some of the others also. We may proceed in various ways; for instance, determine the values of each tenth as compared with one tenth on the silver scale; or compare the successive double tenths [*a*·2], [1·3], [2·4], ... [8·*b*] with a space of two tenths on the silver scale, and the single tenths [*a*·1], [9·*b*] with a space of one tenth on the silver scale; or we may compare the double tenths [*a*·2], [2·4], [4·6], [6·8], [8·*b*] with a two-tenths space on the silver scale, and the three tenths [2·5], [3·6], [4·7], [5·8] with a space of three tenths on the silver scale; or compare the double tenths as in the preceding case, and the four centre tenths [3·4], [4·5], [5·6], [6·7] with a space of one tenth on the silver scale; or, among many other available combinations, the following:—compare the successive double tenths [*a*·2], [2·4], [4·6], [6·8], [8·*b*] with a space of two tenths on the silver scale, and the three tenths [*a*·3], [3·6], [4·7], [7·*b*] with a space of three tenths on the silver scale.

If in each of these cases we investigate the algebraical expressions for the required errors of division, and also the probable errors of such determinations, it will be found that the last named is the best. It has therefore been adopted.

With respect to the subdivisions of [2·3], [6·7] into hundredths, each hundredth has been measured twenty times with each microscope, by which means their errors are very accurately obtained.

2.

On each of the platinum disks which mark the inches, and on the platinum slips which bear the subdivisions of the extreme inches, are traced two parallel longitudinal lines. These form fragments of a pair of parallel straight lines running the length of the bar and defining the part of the lines (transverse) of division which is to be observed. It will be convenient to give a name to that imaginary line which lies on the surface of the scale, parallel to and midway between the parallel lines just referred to. We shall call it the *line of measurements*. So, on the silver scale, those *points* of the transverse division lines which are actually bisected by the microscopes lie in the line of measurements of that scale.

In determining the errors of the inch divisions, the adjustments are the following:— 1. The upper surface of the bar to be truly horizontal. 2. The line of measurements to be parallel to the rails or direction of motion of the carriages. 3. The micrometer microscopes to be placed with their axes vertical, and their foci in the line of measurements on the upper surface of the bar. The distance of the zeros of the microscopes is for convenience made to exceed the quantity under measurement by from 10 to 40 divisions; so that the points will lie in the following order if, for instance, the microscopes are three inches apart, and the points *ad* under H and K respectively

$$z\ a \qquad\qquad\qquad\qquad d\ z'$$

z z' are the zeros of the microscopes; and za is the quantity measured by H, and $z'd$ that measured by K. In this position all measurements are positive; if α be the reading of H and β that of K, h the value of one division of the micrometer in H, k the value of one division of the micrometer in K, then Z being the actual distance of the points z z',

$$Z = [a \cdot d] + \alpha h + \beta k.$$

The method of observing and recording the observations is as follows:—Suppose, for instance, the six-inch spaces ag, gp are being compared. The lines a, g are first brought into view in the microscopes H K respectively, and by the movement of the slow motion screws (transverse) of the carriages the line of measurements is made to pass through the cross-hairs of the two microscopes; the lines are then bisected and readings taken in the following order,—1. one of H, 2. one of K, 3. one of H, 4. one of K, 5. one of H, 6. one of K. Thus we have three readings of a and three of g. The bar is moved six inches to the left, and g, p come into view under the microscopes H, K respectively. The transverse adjustment being made, to cause the line of measurements to pass through the cross-hairs of the two microscopes, the lines g, p are observed in the same manner as were a, g; that is, three readings of each are taken, the microscopes being read alternately. We have now one comparison of ag, gp. At each visit to the Bar Room *three* comparisons of the two spaces are made, involving thirty-six micrometer readings. The spaces are brought under the microscopes in the order

$$ag,\ gp : gp,\ ag : ag,\ gp$$

or the reverse—

$$gp,\ ag : ag,\ gp : gp,\ ag$$

The following is a specimen page of the observation book, being the readings taken at one visit:—

Date. Observer.	Space.	Readings of H	Readings of K	Space.	Readings of H	Readings of K
December 12, 2.30 P.M.	*ag*	14·2 14·0 13·6	19·2 19·3 18·7	*ag*	13·3 13·8 13·4	18·6 18·6 18·8
	gp	14·9 15·0 14·8	18·4 17·4 17·4	*ag*	13·3 13·3 13·7	18·6 18·8 18·6
Quartermaster Steel, R.E.	*gp*	14·5 15·0 14·4	17·3 16·8 17·1	*gp*	14·3 14·0 14·2	16·0 15·7 16·2

Here we have three distinct comparisons of ag, gp, each comparison involving twelve micrometer readings. We may either treat the comparisons separately, or take the mean of all the readings of ag and of gp. Thus, if α be the mean of the nine readings of a under H, β that of the nine readings of g under K, and if α' β' similarly pertain to gp,

$$Z = [a \cdot g] + \alpha h + \beta k$$
$$Z = [g \cdot p] + \alpha' h + \beta' k$$

These equations we shall take to be the result of this visit. The scale being left continually under the microscopes was visited three times a day. The ninety comparisons of the six-inch spaces occupied five days in November 1863 and six days in December. The comparison of the five-inch spaces af, bg, similar in every respect to the above, occupied seven days in November and two in December,—in all fifty-one comparisons. In the comparison of the four-inch spaces, which occupied three days in November and six in December, at each visit two comparisons of the three spaces were made in the order

$$ae,\ bf,\ cg : cg,\ bf,\ ae$$

involving thirty-six micrometer readings. Each visit or page of the observation book gives then this result:

$$Z = [a \cdot e] + \alpha h + \beta k$$
$$Z = [b \cdot f] + \alpha' h + \beta' k$$
$$Z = [c \cdot g] + \alpha'' h + \beta'' k$$

where each of the quantities α α' . . . is the mean of six micrometer readings. There are fifty comparisons in this series.

The comparisons of the three-inch spaces occupied four days in November and four days in December. At each visit were obtained, from forty-eight readings, two comparisons of the four spaces in the following order

$$ad,\ be,\ cf,\ dg : dg,\ cf,\ be,\ ad$$

giving the results—

$$Z = [a \cdot d] + \alpha h + \beta k$$
$$Z = [b \cdot e] + \alpha' h + \beta' k$$
$$Z = [c \cdot f] + \alpha_{,} h + \beta_{,} k$$
$$Z = [d \cdot g] + \alpha'' h + \beta'' k$$

where each of the quantities α α' . . . is the mean of six micrometer readings. There are fifty comparisons in this series.

The total number of micrometer readings from which the errors of division of the lines b c d e f g have to be deduced is 3792.

In applying to these observations the method of least squares, we shall determine errors of divisions which shall make, not the sum of the squares of the errors of the 3792 individual micrometer readings a minimum, but the sum of the squares of the errors of the determinations at each visit a minimum.

Tables I., II., III., IV. contain the results of the observations on the six-inch, five-inch, four-inch, and three-inch spaces in the form explained above: the binomials ranged in a horizontal line in each table being the results of one visit.

Observations for determining the Errors of Inch Divisions on the Foot OF.

Microscopes 6 inches apart.

Table I.

Z − [*a*·*g*]	Z − [*g*·*p*]
− 4·17 *h* + 23·40 *k*	− 3·71 *h* + 20·60 *k*
+ 14·81 *h* + 19·30 *k*	+ 16·29 *h* + 14·70 *k*
+ 19·37 *h* + 14·91 *k*	+ 19·31 *h* + 12·84 *k*
24·41 *h* + 8·72 *k*	25·19 *h* + 6·41 *k*
13·97 *h* + 18·08 *k*	11·57 *h* + 16·85 *k*
14·12 *h* + 18·12 *k*	13·70 *h* + 16·69 *k*
12·73 *h* + 19·70 *k*	13·01 *h* + 17·58 *k*
14·88 *h* + 15·78 *k*	15·03 *h* + 12·89 *k*
14·04 *h* + 15·23 *k*	14·69 *h* + 12·21 *k*
15·16 *h* + 11·50 *k*	14·41 *h* + 9·21 *k*
19·13 *h* + 10·69 *k*	19·98 *h* + 7·63 *k*
19·47 *h* + 15·91 *k*	19·92 *h* + 13·80 *k*
15·78 *h* + 18·09 *k*	14·45 *h* + 17·77 *k*
16·48 *h* + 17·76 *k*	16·09 *h* + 15·94 *k*
11·92 *h* + 22·06 *k*	11·99 *h* + 20·40 *k*
16·94 *h* + 17·67 *k*	15·00 *h* + 17·11 *k*
15·24 *h* + 18·55 *k*	14·93 *h* + 15·82 *k*
13·62 *h* + 18·80 *k*	14·57 *h* + 16·92 *k*
16·67 *h* + 17·40 *k*	15·08 *h* + 15·27 *k*
15·40 *h* + 19·32 *k*	15·30 *h* + 17·83 *k*
15·77 *h* + 17·71 *k*	13·09 *h* + 18·01 *k*
17·37 *h* + 16·33 *k*	20·19 *h* + 11·98 *k*
16·54 *h* + 17·63 *k*	15·27 *h* + 16·28 *k*
19·30 *h* + 16·54 *k*	19·69 *h* + 13·67 *k*
16·38 *h* + 18·61 *k*	15·72 *h* + 15·99 *k*
17·17 *h* + 16·46 *k*	16·67 *h* + 14·87 *k*
20·43 *h* + 14·10 *k*	19·80 *h* + 12·98 *k*
16·21 *h* + 16·12 *k*	16·79 *h* + 14·11 *k*
16·42 *h* + 15·89 *k*	16·32 *h* + 14·34 *k*
+ 19·07 *h* + 14·11 *k*	+ 17·83 *h* + 14·80 *k*
+ 15·82 *h* + 16·82 *k*	+ 15·61 *h* + 14·85 *k*

N.B.—The last detached line in each of these tables is the mean of the quantities in the column at the bottom of which it stands.

Microscopes 5 inches apart.

TABLE II.

Z − [a.f]	Z − [b.g]
11·95 h + 16·61 k	17·60 h + 15·76 k
10·66 h + 13·82 k	9·24 h + 19·39 k
16·41 h + 10·16 k	19·88 h + 10·55 k
19·42 h + 7·36 k	20·64 h + 11·10 k
13·96 h + 13·77 k	13·13 h + 18·12 k
17·44 h + 9·00 k	17·07 h + 14·02 k
16·97 h + 11·45 k	16·68 h + 15·63 k
16·67 h + 11·36 k	15·40 h + 16·39 k
14·73 h + 13·60 k	15·92 h + 16·22 k
15·92 h + 8·69 k	15·91 h + 13·11 k
16·54 h + 4·99 k	14·09 h + 11·83 k
15·39 h + 16·04 k	16·12 h + 18·03 k
17·22 h + 15·07 k	16·86 h + 18·62 k
16·39 h + 15·64 k	18·07 h + 17·04 k
16·11 h + 16·51 k	15·69 h + 20·81 k
15·96 h + 15·35 k	17·98 h + 16·65 k
15·92 h + 15·26 k	18·33 h + 16·09 k
15·74 h + 12·63 k	16·39 h + 15·84 k

Microscopes 4 inches apart.

TABLE III.

Z − [a.e]	Z − [b.f]	Z − [c.g]
19·83 h + 21·65 k	18·66 h + 20·60 k	20·50 h + 24·80 k
19·02 h + 21·02 k	20·53 h + 17·60 k	19·48 h + 24·25 k
23·82 h + 19·12 k	22·37 h + 18·43 k	24·28 h + 22·70 k
25·30 h + 18·40 k	27·63 h + 14·33 k	25·60 h + 23·10 k
29·20 h + 14·65 k	28·88 h + 13·43 k	29·45 h + 18·85 k
24·15 h + 19·85 k	18·78 h + 23·55 k	19·22 h + 29·16 k
21·50 h + 23·68 k	20·80 h + 23·17 k	23·55 h + 25·42 k
28·05 h + 15·48 k	23·75 h + 18·85 k	22·47 h + 26·22 k
24·40 h + 18·37 k	23·60 h + 18·42 k	25·15 h + 21·20 k
23·53 h + 21·05 k	22·77 h + 20·73 k	23·98 h + 24·68 k
21·40 h + 25·32 k	21·33 h + 24·40 k	21·40 h + 30·02 k
23·63 h + 23·92 k	22·85 h + 23·27 k	24·46 h + 27·85 k
22·85 h + 25·75 k	22·66 h + 25·05 k	23·00 h + 30·06 k
20·55 h + 28·20 k	20·98 h + 26·68 k	20·52 h + 33·13 k
22·11 h + 25·88 k	21·18 h + 25·35 k	24·10 h + 28·45 k
23·15 h + 23·75 k	22·46 h + 23·22 k	23·90 h + 27·50 k
20·62 h + 28·72 k	20·63 h + 27·27 k	21·05 h + 33·03 k
24·18 h + 23·86 k	23·38 h + 22·97 k	27·43 h + 25·62 k
24·08 h + 25·25 k	25·25 h + 23·28 k	26·13 h + 28·22 k
24·18 h + 24·61 k	23·33 h + 23·03 k	26·62 h + 26·40 k
26·83 h + 22·15 k	27·03 h + 20·96 k	26·96 h + 26·95 k
24·33 h + 23·16 k	24·92 h + 21·38 k	27·67 h + 24·85 k
24·95 h + 24·02 k	24·32 h + 22·38 k	25·82 h + 27·30 k
23·36 h + 23·65 k	22·46 h + 23·45 k	27·43 h + 24·47 k
24·95 h + 24·27 k	24·55 h + 23·30 k	25·60 h + 28·18 k
23·60 h + 22·63 k	23·00 h + 21·80 k	24·23 h + 26·50 k

Microscopes 3 inches apart.

TABLE IV.

Z – [a·d]	Z – [b·e]	Z – [c·f]	Z – [d·g]
23·83 h + 10·06 k	23·87 h + 10·95 k	23·95 h + 10·42 k	25·25 h + 12·20 k
21·77 h + 11·18 k	21·57 h + 12·92 k	21·75 h + 11·95 k	21·88 h + 15·00 k
22·53 h + 10·80 k	21·72 h + 12·83 k	20·53 h + 14·03 k	23·55 h + 15·27 k
22·42 h + 10·75 k	22·88 h + 11·38 k	22·30 h + 12·42 k	21·46 h + 15·88 k
23·35 h + 10·63 k	23·67 h + 11·32 k	20·22 h + 14·43 k	22·22 h + 16·82 k
21·63 h + 11·83 k	20·92 h + 14·35 k	19·65 h + 14·50 k	19·98 h + 17·68 k
17·40 h + 16·60 k	17·18 h + 17·28 k	16·82 h + 18·48 k	16·43 h + 21·98 k
21·33 h + 12·42 k	19·30 h + 15·65 k	19·80 h + 14·32 k	21·27 h + 16·78 k
21·07 h + 13·86 k	22·42 h + 13·50 k	21·83 h + 13·30 k	24·98 h + 13·65 k
18·42 h + 15·05 k	17·28 h + 16·88 k	17·52 h + 16·55 k	16·36 h + 21·00 k
15·38 h + 18·62 k	14·80 h + 20·53 k	12·13 h + 23·82 k	14·72 h + 23·73 k
15·65 h + 16·86 k	15·00 h + 18·46 k	16·20 h + 17·47 k	17·78 h + 20·33 k
17·53 h + 16·35 k	17·72 h + 17·11 k	18·80 h + 17·00 k	19·33 h + 19·98 k
19·83 h + 13·83 k	18·38 h + 16·27 k	18·13 h + 16·62 k	18·62 h + 19·75 k
17·82 h + 16·68 k	18·22 h + 17·08 k	17·92 h + 17·80 k	18·11 h + 20·37 k
15·23 h + 18·33 k	15·98 h + 18·65 k	16·66 h + 17·68 k	19·27 h + 19·08 k
15·93 h + 14·48 k	15·75 h + 15·28 k	16·70 h + 15·25 k	17·38 h + 17·55 k
15·03 h + 14·87 k	15·37 h + 15·48 k	16·08 h + 14·47 k	13·98 h + 21·15 k
17·72 h + 12·63 k	18·78 h + 12·05 k	19·48 h + 12·50 k	21·22 h + 13·25 k
13·92 h + 15·76 k	15·08 h + 14·90 k	13·17 h + 17·07 k	13·48 h + 20·37 k
15·36 h + 15·35 k	14·13 h + 15·88 k	14·82 h + 16·62 k	16·53 h + 18·13 k
15·72 h + 14·96 k	14·45 h + 17·25 k	13·76 h + 17·53 k	18·12 h + 16·42 k
21·52 h + 21·73 k	21·10 h + 23·23 k	20·77 h + 24·47 k	21·05 h + 26·90 k
20·16 h + 23·70 k	19·56 h + 23·96 k	20·06 h + 24·05 k	23·78 h + 23·95 k
24·38 h + 20·06 k	24·47 h + 21·05 k	24·65 h + 20·68 k	25·05 h + 23·16 k
19·00 h + 15·10 k	18·78 h + 16·17 k	18·55 h + 16·54 k	19·67 h + 18·82 k

Let x_g be the error in position of the line g, which is supposed to be equidistant from a and p, so that if the length of the whole scale, or $[a \cdot p]$, be $= 12$ I,

$$[a \cdot g] = 6\,\mathrm{I} + x_g \qquad (1)$$
$$[g \cdot p] = 6\,\mathrm{I} - x_g$$

Again, let $x_b\ x_c\ x_d\ x_e\ x_f$ be the errors of the lines $b\ c\ d\ e\ f$ considered as subdividing $[a \cdot g]$ into six equal parts, so that

$$[a \cdot b] = \tfrac{1}{6}\,[a \cdot g] + x_b = \mathrm{I} + x_b + \tfrac{1}{6}\,x_g \qquad (2)$$
$$[a \cdot c] = \tfrac{2}{6}\,[a \cdot g] + x_c = 2\,\mathrm{I} + x_c + \tfrac{2}{6}\,x_g$$
$$[a \cdot d] = \tfrac{3}{6}\,[a \cdot g] + x_d = 3\,\mathrm{I} + x_d + \tfrac{3}{6}\,x_g$$
$$[a \cdot e] = \tfrac{4}{6}\,[a \cdot g] + x_e = 4\,\mathrm{I} + x_e + \tfrac{4}{6}\,x_g$$
$$[a \cdot f] = \tfrac{5}{6}\,[a \cdot g] + x_f = 5\,\mathrm{I} + x_f + \tfrac{5}{6}\,x_g$$

And again, let $x_2\, x_3\, x_4\, x_6\, x_7\, x_8$ be the errors of the lines 2 3 4 6 7 8 with reference to the lines a and b; that is,

$$\begin{aligned}
[a\cdot 2] &= \tfrac{2}{10}\,[a\cdot b] + x_2 = \tfrac{2}{10}\,\mathrm{I} + x_2 + \tfrac{2}{10}\,x_b + \tfrac{2}{60}\,x_g \\
[a\cdot 3] &= \tfrac{3}{10}\,[a\cdot b] + x_3 = \tfrac{3}{10}\,\mathrm{I} + x_3 + \tfrac{3}{10}\,x_b + \tfrac{3}{60}\,x_g \\
[a\cdot 4] &= \tfrac{4}{10}\,[a\cdot b] + x_4 = \tfrac{4}{10}\,\mathrm{I} + x_4 + \tfrac{4}{10}\,x_b + \tfrac{4}{60}\,x_g \\
[a\cdot 6] &= \tfrac{6}{10}\,[a\cdot b] + x_6 = \tfrac{6}{10}\,\mathrm{I} + x_6 + \tfrac{6}{10}\,x_b + \tfrac{6}{60}\,x_g \\
[a\cdot 7] &= \tfrac{7}{10}\,[a\cdot b] + x_7 = \tfrac{7}{10}\,\mathrm{I} + x_7 + \tfrac{7}{10}\,x_b + \tfrac{7}{60}\,x_g \\
[a\cdot 8] &= \tfrac{8}{10}\,[a\cdot b] + x_8 = \tfrac{8}{10}\,\mathrm{I} + x_8 + \tfrac{8}{10}\,x_b + \tfrac{8}{60}\,x_g
\end{aligned} \tag{3}$$

The value of x_g results immediately from the means of the columns in Table I.

$$\begin{aligned}
[a\cdot g] - [g\cdot p] &= [15\cdot 61\,h + 14\cdot 85\,k] - [15\cdot 82\,h + 16\cdot 82\,k] \\
2\,x_g &= -\,0\cdot 21\,h - 1\cdot 97\,k \\
x_g &= -\,0\cdot 10\,h - 0\cdot 99\,k
\end{aligned} \tag{4}$$

It is convenient to remember that h and k are almost exactly equal to one another and to $\tfrac{8}{10}$ of the millionth of a yard. We shall obtain their exact values further on.

In Tables I., II. each numerical quantity results from the mean of 9 micrometer readings; if the probable error of a single reading be ϵ, that of each binomial in these Tables is $\tfrac{1}{3}\,\epsilon\,\sqrt{2}$. Similarly, in Tables III., IV. the probable error of each binomial is $\tfrac{1}{3}\,\epsilon\,\sqrt{3}$. The weight, therefore, of the quantities in Tables I., II. is greater than that of the quantities in Tables III., IV. in the proportion of 3 : 2. In fact the 17 lines in Table II. result from 51 comparisons, but the 25 lines in Tables III. or IV. from 50 comparisons.

In Table IV. let the quantities in the n^{th} line be

$$\alpha_n\ \beta_n\ \gamma_n\ \delta_n$$

The corresponding distance of the zeros of the microscopes

$$3\,\mathrm{I} - z_n.$$

In Table III., similarly,

$$\alpha'_n\ \beta'_n\ \gamma'_n\quad,\quad 4\,\mathrm{I} - z'_n.$$

In Table II., similarly,

$$\alpha''_n\ \beta''_n\ \gamma''_n\quad,\quad 5\,\mathrm{I} - z''_n.$$

Then the equations will be—

$$\begin{aligned}
z_1 \qquad\ \ &+ x_a + \alpha_1 = 0 \\
z_1 - x_b &+ x_e + \beta_1 = 0 \\
z_1 - x_c &+ x_f + \gamma_1 = 0 \\
z_1 - x_d &\qquad\ + \delta_1 = 0 \\
z_2 \qquad\ \ &+ x_a + \alpha_2 = 0 \\
z_2 - x_b &+ x_e + \beta_2 = 0 \\
z_2 - x_c &+ x_f + \gamma_2 = 0 \\
z_2 - x_d &\qquad\ + \delta_2 = 0 \\
&\ \ \vdots \qquad\qquad \vdots
\end{aligned} \tag{5}$$

$$
\begin{array}{l}
z'_1 \qquad + x_e + \alpha'_1 = 0 \\
z'_1 - x_b + x_f + \beta'_1 = 0 \\
z'_1 - x_c \qquad + \gamma'_1 = 0 \\
z'_2 \qquad + x_e + \alpha'_2 = 0 \\
z'_2 - x_b + x_f + \beta'_2 = 0 \\
z'_2 - x_c \qquad + \gamma'_2 = 0 \\
\cdot \qquad\qquad \cdot \\
\cdot \qquad\qquad \cdot \\
\cdot \qquad\qquad \cdot \\
\cdot \qquad\qquad \cdot \\
\cdot \qquad\qquad \cdot \\
z''_1 \qquad + x_f + \alpha''_1 = 0 \\
z''_1 - x_b \qquad + \beta''_1 = 0 \\
z''_2 \qquad + x_f + \alpha''_2 = 0 \\
z''_2 - x_b \qquad + \beta''_2 = 0 \\
\cdot \qquad\qquad \cdot \\
\cdot \qquad\qquad \cdot \\
\cdot \qquad\qquad \cdot \\
\cdot \qquad\qquad \cdot
\end{array}
$$

The last equations, namely, those in z'', have a greater weight than the others, as has been explained. They must, therefore, be multiplied by $\sqrt{\frac{3}{2}}$, or, which will be sufficiently near, by the square root of $\frac{25}{17}$; this will considerably simplify the solution. Solving by the method of least squares—

$$
\begin{array}{l}
4 z_1 - x_b - x_c + x_e + x_f + \alpha_1 + \beta_1 + \gamma_1 + \delta_1 = 0 \\
4 z_2 - x_b - x_c + x_e + x_f + \alpha_2 + \beta_2 + \gamma_2 + \delta_2 = 0 \\
4 z_3 - x_b - x_c + x_e + x_f + \alpha_3 + \beta_3 + \gamma_3 + \delta_3 = 0 \\
\cdot \\
\cdot \\
\cdot \\
\cdot \\
3 z'_1 - x_b - x_c + x_e + x_f + \alpha'_1 + \beta'_1 + \gamma'_1 = 0 \\
3 z'_2 - x_b - x_c + x_e + x_f + \alpha'_2 + \beta'_2 + \gamma'_2 = 0 \\
3 z'_3 - x_b - x_c + x_e + x_f + \alpha'_3 + \beta'_3 + \gamma'_3 = 0 \\
\cdot \\
\cdot \\
\cdot \\
2 z''_1 - x_b + x_f + \alpha''_1 + \beta''_1 = 0 \\
2 z''_2 - x_b + x_f + \alpha''_2 + \beta''_2 = 0 \\
2 z''_3 - x_b + x_f + \alpha''_3 + \beta''_3 = 0 \\
-(z) - (z') - \tfrac{25}{17}(z'') + 3\,nx_b - nx_e - nx_f - (\beta) - (\beta') - \tfrac{25}{17}(\beta\ ) = 0 \\
-(z) - (z') + 2\,nx_c - nx_f - (\gamma) - (\gamma') = 0 \\
2\,nx_d + (\alpha) - (\delta) = 0 \\
(z) + (z') - nx_b + 2\,nx_e + (\beta) + (\alpha') = 0 \\
(z) + (z') + \tfrac{25}{17}(z'') - nx_b - nx_c + 3nx + (\gamma) + (\beta') + \tfrac{25}{17}(\alpha'') = 0
\end{array}
$$

Divide through each of these last equations by n, which represents 25, and if z represent the *mean* of the quantities $z_1\, z_2 \ldots z_{25}$; α the mean of $\alpha_1\, \alpha_2 \ldots \alpha_{25}$, and so on, we get,

$$\begin{array}{l}
4z - x_b - x_c + x_e + x_f + \alpha + \beta + \gamma + \delta = 0 \\
3z' - x_b - x_c + x_e + x_f + \alpha' + \beta' + \gamma' = 0 \\
2z'' - x_b + x_f + \alpha'' + \beta'' = 0 \\
-z - z' - z'' + 3x_b - x_e - x_f - \beta - \beta' - \beta'' = 0 \\
-z - z' + 2x_c - x_f - \gamma - \gamma' = 0 \\
2x_d + \alpha - \delta = 0 \\
z + z' - x_b + 2x_e + \beta + \alpha' = 0 \\
z + z' + z'' - x_b - x_c + 3x_f + \gamma + \beta' + \alpha'' = 0
\end{array}$$

Write now A, B, C, D, E, F, G, H for the absolute terms of these eight equations, and we get, eliminating $z\, z'\, z''$, the following expressions for $x_b \ldots x_g$:

$$\begin{array}{l}
0 = 18x_b + 3A + 4B + 5C + 11D + 4E + 2G + H \\
0 = 18x_c + 6A + 8B + C + 4D + 17E - 5G + 2H \\
0 = 18x_d + 9F \\
0 = 18x_e - 6A - 8B - C + 2D - 5E + 17G + 4H \\
0 = 18x_f - 3A - 4B - 5C + D + 2E + 4G + 11H
\end{array} \qquad (6)$$

And the weights of the determinations are:

$$\begin{array}{lll}
x_b & \text{weight} & \frac{25 \cdot 18}{11} \\
x_c & \text{"} & \frac{25 \cdot 18}{17} \\
x_d & \text{"} & \frac{25 \cdot 18}{9} \\
x_e & \text{"} & \frac{25 \cdot 18}{17} \\
x_f & \text{"} & \frac{25 \cdot 18}{11}
\end{array} \qquad (7)$$

If now we substitute for A B C . . . H their values in terms of $\alpha\, \beta \ldots$, we get

$$\begin{array}{l}
0 = 6x_b + \alpha - 2\beta + 0 + \delta + 2\alpha' - 2\beta' + 0 + 2\alpha'' - 2\beta'' \\
0 = 6x_c + 2\alpha - \beta - 3\gamma + 2\delta + \alpha' + 2\beta' - 3\gamma' + \alpha'' - \beta'' \\
0 = 6x_d + 3\alpha - 3\delta \\
0 = 6x_e - 2\alpha + 3\beta + \gamma - 2\delta + 3\alpha' - 2\beta' - \gamma' + \alpha'' - \beta'' \\
0 = 6x_f - \alpha - 0 + 2\gamma - \delta + 0 + 2\beta' - 2\gamma' + 2\alpha'' - 2\beta''
\end{array} \qquad (8)$$

Here the quantities sought are expressed in terms of known quantities: and from the Tables we find

$$\begin{aligned} \alpha &= 19{\cdot}00\,h + 15{\cdot}10\,k \\ \beta &= 18{\cdot}78\,h + 16{\cdot}17\,k \\ \gamma &= 18{\cdot}55\,h + 16{\cdot}54\,k \\ \delta &= 19{\cdot}67\,h + 18{\cdot}82\,k \\ \alpha' &= 23{\cdot}60\,h + 22{\cdot}63\,k \\ \beta' &= 23{\cdot}00\,h + 21{\cdot}80\,k \\ \gamma' &= 24{\cdot}23\,h + 26{\cdot}50\,k \\ \alpha'' &= 15{\cdot}74\,h + 12{\cdot}63\,k \\ \beta'' &= 16{\cdot}39\,h + 15{\cdot}84\,k \end{aligned} \qquad (9)$$

Whence,

$$\begin{aligned} x_b &= -\,0{\cdot}17\,h + 0{\cdot}53\,k \\ x_c &= +\,0{\cdot}14\,h + 2{\cdot}41\,k \\ x_d &= +\,0{\cdot}33\,h + 1{\cdot}86\,k \\ x_e &= +\,0{\cdot}42\,h + 1{\cdot}37\,k \\ x_f &= +\,0{\cdot}89\,h + 2{\cdot}78\,k \end{aligned} \qquad (10)$$

3.

The silver scale and **OF** being mounted for comparison, it is essential that the *line of measurements* in the one shall form one and the same straight line with the *line of measurements* on the other: further, that this line, being truly horizontal and parallel to the line of rails, shall appear, as viewed in the microscopes, to intersect the cross-hairs both in H and K.

While the left-hand line, that marked o, on the silver scale is in the zero of the micrometer H; by gradually moving F on its rollers, the lines *a* 2 4 6 8 on **OF** can be brought in succession into the zero of K. At each visit to the Bar Room each double tenth on **OF** (five in number) was compared with [0·2].

First, o of the silver scale under H and *a* of **OF** under K are read, giving

$$\alpha_1\, h + \beta_1\, k$$

then by the motion of the carriages two tenths of an inch to the left, 2 of the silver scale and 2 of **OF** are brought under H and K; these lines being read, we have

$$\alpha'_1\, h + \beta'_1\, k$$

Secondly, o of the silver scale is brought under H and 2 of **OF** under K; the lines being bisected and read, give

$$\alpha_2\, h + \beta_2\, k$$

then by the motion of the carriages two tenths of an inch to the left, 2 of the silver scale and 4 of **OF** are brought under H and K; these lines being read, we get

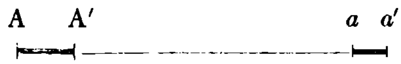
$$\alpha'_2\, h + \beta'_2\, k$$

Thirdly, o of the silver scale and 4 of **OF** are brought under the microscopes and read, and so on. The order of observing is, in short, this :—

	H	K
	Silver Scale.	**OF**
Left lines	o	a
Right lines	2	2
Left	o	2
Right	2	4
Left	o	4
Right	2	6
Left	o	6
Right	2	8
Left	o	8
Right	2	b

there being taken in each case three readings of each microscope for each line, as explained before. The results are given in detail in Tables V. and VI.

Let S be the length of the silver scale [o·2], and S′ = [a·2] on **OF**.



In the diagram let AA′ = S, aa' = S′. Let Aa = q : the distance of the microscopes = $q + z$; then, when the left-hand lines are under the microscopes, we get

$$q + \alpha_1 h + \beta_1 k = q + z$$
$$\alpha_1 h + \beta_1 k - z = 0 ;$$

and when the right lines are under the microscopes, we have

$$q + S' - S + \alpha'_1 h + \beta'_1 k = q + z$$
$$S' - S + \alpha'_1 h + \beta'_1 k - z = 0$$
$$\therefore S' - S + (\alpha'_1 - \alpha_1) h + (\beta'_1 - \beta_1) k = 0$$
$$\therefore [0 \cdot 2] - [a \cdot 2] = (\alpha'_1 - \alpha_1) h + (\beta'_1 - \beta_1) k$$

Similarly,

$$[0 \cdot 2] - [2 \cdot 4] = (\alpha'_2 - \alpha_2) h + (\beta'_2 - \beta_2) k$$
$$[0 \cdot 2] - [4 \cdot 6] = (\alpha'_3 - \alpha_3) h + (\beta'_3 - \beta_3) k$$
$$[0 \cdot 2] - [6 \cdot 8] = (\alpha'_4 - \alpha_4) h + (\beta'_4 - \beta_4) k$$
$$[0 \cdot 2] - [8 \cdot b] = (\alpha'_5 - \alpha_5) h + (\beta'_5 - \beta_5) k$$

The comparisons of [o·3] on the silver scale with [a·3] [3·6] [4·7] [7·b] are similarly effected.

Table V.

Observations for determining Errors of Tenths of Inches on **OF**.

Comparisons with 0 – 2 of Silver Scale.

[a·2]		[2·4]		[4·6]		[6·8]		[8·b]	
Left Lines.	Right Lines.	Left Lines.	Right Lines.	Left Lines.	Right Lines.	Left Lines.	Right Lines.	Left Lines.	Right Lines.
15·2 h + 9·1 k	16·1 h + 17·0 k	16·1 h + 17·2 k	16·8 h + 35·6 k	16·8 h + 34·8 k	16·4 h + 43·8 k	14·3 h + 44·6 k	15·1 h + 59·3 k	15·6 h + 14·6 k	14·4 h + 30·8 k
7·0 h + 12·2 k	8·8 h + 18·0 k	11·1 h + 7·6 k	11·7 h + 25·9 k	11·1 h + 8·3 k	10·9 h + 16·2 k	10·2 h + 18·8 k	8·7 h + 33·6 k	16·1 h + 7·3 k	15·5 h + 21·2 k
17·0 h + 15·2 k	17·1 h + 23·1 k	19·2 h + 15·1 k	17·7 h + 34·8 k	15·4 h + 15·1 k	15·1 h + 23·3 k	16·0 h + 15·1 k	15·6 h + 29·2 k	15·5 h + 13·5 k	14·7 h + 29·0 k
16·0 h + 13·0 k	16·5 h + 19·6 k	17·0 h + 7·7 k	16·7 h + 25·3 k	15·9 h + 10·6 k	11·3 h + 21·3 k	10·5 h + 9·2 k	11·2 h + 22·5 k	12·2 h + 1·3 k	8·5 h + 20·3 k
15·5 h + 21·7 k	14·7 h + 29·8 k	13·5 h + 16·7 k	13·9 h + 35·4 k	15·3 h + 12·2 k	16·4 h + 19·3 k	18·2 h + 16·2 k	17·6 h + 32·6 k	15·8 h + 11·1 k	19·1 h + 22·9 k
20·0 h + 28·6 k	16·0 h + 39·9 k	12·2 h + 2·7 k	9·2 h + 25·0 k	6·1 h + 7·4 k	11·9 h + 9·7 k	15·3 h + 3·5 k	20·2 h + 14·0 k	18·1 h + 13·0 k	13·8 h + 32·2 k
15·8 h + 23·0 k	16·7 h + 30·5 k	15·4 h + 20·0 k	15·0 h + 39·2 k	13·1 h + 19·1 k	19·2 h + 21·2 k	13·7 h + 16·1 k	11·9 h + 32·1 k	13·5 h + 23·4 k	16·1 h + 35·7 k
14·2 h + 7·7 k	15·5 h + 14·2 k	13·8 h + 1·3 k	18·0 h + 15·4 k	12·6 h + 7·2 k	13·8 h + 13·7 k	8·7 h + 4·1 k	9·5 h + 17·2 k	5·8 h + 8·4 k	11·6 h + 17·1 k
18·0 h + 20·1 k	23·1 h + 23·3 k	20·8 h + 18·1 k	23·1 h + 35·0 k	15·7 h + 16·8 k	19·9 h + 21·4 k	17·3 h + 14·4 k	21·7 h + 25·6 k	15·6 h + 17·0 k	14·2 h + 32·2 k
19·1 h + 13·3 k	25·4 h + 15·6 k	23·9 h + 9·5 k	27·1 h + 24·4 k	23·9 h + 12·1 k	28·1 h + 17·1 k	18·6 h + 0·8 k	20·2 h + 12·4 k	18·5 h + 20·8 k	21·3 h + 33·2 k
10·5 h + 16·0 k	12·4 h + 22·6 k	15·0 h + 9·2 k	12·8 h + 30·1 k	9·9 h + 18·9 k	18·4 h + 20·3 k	16·0 h + 16·9 k	16·5 h + 30·9 k	16·9 h + 14·1 k	15·2 h + 30·5 k
13·7 h + 13·8 k	14·1 h + 21·0 k	13·9 h + 1·0 k	15·9 h + 16·5 k	13·6 h + 1·5 k	16·4 h + 7·2 k	15·8 h + 5·3 k	16·6 h + 16·6 k	8·1 h + 14·9 k	8·1 h + 29·5 k
15·1 h + 19·4 k	0·5 h + 42·3 k	4·1 h + 5·9 k	11·5 h + 16·9 k	10·9 h + 20·0 k	8·8 h + 30·9 k	12·2 h + 9·5 k	9·2 h + 26·7 k	7·6 h + 17·0 k	13·5 h + 26·8 k
10·2 h + 17·3 k	15·0 h + 21·1 k	14·9 h + 15·5 k	14·7 h + 34·1 k	17·4 h + 15·7 k	19·7 h + 22·3 k	16·1 h + 15·6 k	16·1 h + 30·3 k	18·9 h + 19·5 k	20·0 h + 32·1 k
10·1 h + 19·9 k	8·6 h + 28·8 k	7·6 h + 4·8 k	10·5 h + 21·1 k	5·8 h + 10·0 k	11·3 h + 11·7 k	8·5 h + 11·1 k	7·1 h + 27·4 k	5·2 h + 3·6 k	5·2 h + 18·0 k
16·9 h + 6·0 k	15·5 h + 16·4 k	21·2 h + 19·3 k	16·6 h + 40·9 k	30·2 h + 17·5 k	33·0 h + 23·4 k	21·3 h + 21·2 k	24·1 h + 31·9 k	26·5 h + 13·6 k	23·5 h + 30·5 k
12·1 h + 16·5 k	18·2 h + 18·3 k	20·9 h + 22·3 k	21·6 h + 39·4 k	21·8 h + 15·5 k	25·1 h + 20·3 k	17·3 h + 20·1 k	23·2 h + 27·9 k	13·2 h + 18·8 k	14·0 h + 33·0 k
6·0 h + 4·7 k	16·1 h + 3·2 k	5·4 h + 5·5 k	15·6 h + 14·1 k	2·4 h + 16·5 k	5·5 h + 21·6 k	12·9 h + 29·5 k	4·8 h + 52·3 k	6·3 h + 19·6 k	8·3 h + 31·1 k
4·2 h + 15·5 k	3·7 h + 24·2 k	18·7 h + 17·6 k	29·2 h + 25·4 k	17·7 h + 19·4 k	20·2 h + 24·6 k	25·5 h + 7·5 k	24·1 h + 24·1 k	18·7 h + 9·6 k	21·5 h + 20·9 k
14·6 h + 12·1 k	14·9 h + 20·2 k	9·0 h + 15·5 k	16·4 h + 27·5 k	8·5 h + 13·5 k	16·6 h + 12·2 k	9·6 h + 5·4 k	15·7 h + 14·5 k	7·8 h + 3·6 k	8·1 h + 18·3 k
15·1 h + 13·8 k	11·6 h + 26·4 k	15·2 h + 21·4 k	12·6 h + 43·5 k	5·0 h + 28·5 k	8·3 h + 33·4 k	8·1 h + 14·5 k	14·0 h + 23·5 k	4·7 h + 13·8 k	10·7 h + 23·4 k
19·5 h + 16·2 k	17·6 h + 27·5 k	20·7 h + 21·6 k	20·4 h + 39·7 k	20·6 h + 17·6 k	24·9 h + 21·5 k	15·1 h + 12·5 k	14·6 h + 27·9 k	17·5 h + 7·3 k	22·0 h + 16·9 k
13·1 h + 16·2 k	11·5 h + 25·8 k	17·0 h + 18·9 k	11·3 h + 43·3 k	15·2 h + 17·6 k	7·9 h + 32·6 k	9·0 h + 10·4 k	15·6 h + 18·6 k	9·6 h + 5·4 k	17·5 h + 12·5 k
19·5 h + 13·3 k	21·5 h + 19·8 k	21·3 h + 20·1 k	32·1 h + 28·1 k	10·5 h + 10·6 k	12·4 h + 16·4 k	6·5 h + 8·5 k	13·6 h + 16·1 k	8·4 h + 7·5 k	11·5 h + 19·6 k
11·9 h + 15·0 k	15·2 h + 20·5 k	19·8 h + 14·9 k	22·0 h + 31·5 k	7·4 h + 5·5 k	12·0 h + 7·8 k	8·6 h + 13·8 k	9·6 h + 27·4 k	8·6 h + 28·2 k	32·9 h + 18·9 k
14·01 h + 15·18 k	14·65 h + 22·76 k	15·51 h + 13·18 k	17·30 h + 29·92 k	13·71 h + 14·88 k	16·14 h + 20·53 k	13·81 h + 13·78 k	15·06 h + 26·98 k	12·99 h + 13·08 k	15·25 h + 25·46 k

Table VI.

Observations for determining Errors of Tenths of Inches on **OF**.

Comparisons with o—3 of Silver Scale.

[a·3]		[3·6]		[4·7]		[7·b]	
Left Lines.	Right Lines.	Left Lines.	Right Lines.	Left Lines.	Right Lines.	Left Lines.	Right Lines.
12·8 h + 2·3 k	14·0 h + 11·9 k	14·0 h + 9·3 k	14·3 h + 14·7 k	14·4 h + 11·6 k	14·4 h + 18·3 k	29·8 h + 17·4 k	29·3 h + 29·4 k
14·7 h + 20·9 k	16·2 h + 29·3 k	17·4 h + 19·6 k	16·3 h + 27·1 k	15·4 h + 16·6 k	17·2 h + 22·5 k	13·2 h + 4·8 k	20·5 h + 8·9 k
17·3 h + 20·1 k	19·5 h + 27·5 k	20·6 h + 7·8 k	24·3 h + 9·7 k	23·9 h + 16·0 k	25·7 h + 21·8 k	25·1 h + 16·1 k	25·8 h + 27·3 k
24·1 h + 21·5 k	26·0 h + 29·9 k	25·2 h + 11·6 k	36·5 h + 6·3 k	14·9 h + 15·6 k	9·6 h + 27·6 k	28·5 h + 15·1 k	29·6 h + 27·3 k
21·9 h + 12·7 k	15·1 h + 29·0 k	17·0 h + 13·8 k	25·1 h + 11·2 k	17·9 h + 16·7 k	27·0 h + 15·0 k	17·1 h + 11·5 k	16·5 h + 23·7 k
8·0 h + 13·5 k	12·5 h + 20·5 k	7·8 h + 11·3 k	13·5 h + 10·6 k	13·8 h + 8·7 k	23·4 h + 5·0 k	24·4 h + 3·1 k	23·7 h + 14·0 k
12·4 h + 12·5 k	7·2 h + 27·4 k	1·2 h + 17·5 k	9·1 h + 15·6 k	22·6 h + 12·5 k	27·5 h + 14·6 k	33·5 h + 10·9 k	41·7 h + 14·7 k
19·5 h + 11·9 k	20·0 h + 20·0 k	16·5 h + 15·0 k	26·6 h + 10·1 k	20·8 h + 13·9 k	30·6 h + 11·0 k	16·6 h + 16·7 k	21·2 h + 23·6 k
27·0 h + 37·6 k	24·0 h + 49·0 k	16·6 h + 7·0 k	17·0 h + 12·6 k	14·5 h + 15·5 k	13·0 h + 24·4 k	12·5 h + 11·1 k	13·5 h + 21·3 k
24·0 h + 24·8 k	19·2 h + 38·5 k	13·7 h + 21·3 k	16·7 h + 24·0 k	16·4 h + 3·4 k	9·3 h + 17·5 k	17·5 h + 3·5 k	13·2 h + 19·8 k
16·1 h + 16·7 k	13·5 h + 29·1 k	18·4 h + 24·6 k	24·9 h + 22·9 k	10·9 h + 11·5 k	10·0 h + 19·7 k	14·9 h + 14·9 k	21·2 h + 20·4 k
20·1 h + 11·4 k	22·1 h + 17·8 k	23·6 h + 8·2 k	26·7 h + 11·7 k	14·3 h + 11·1 k	8·0 h + 23·5 k	16·7 h + 19·3 k	20·8 h + 27·2 k
14·2 h + 15·5 k	7·4 h + 32·2 k	11·8 h + 14·5 k	16·0 h + 15·7 k	20·4 h + 13·8 k	19·0 h + 22·2 k	13·5 h + 21·5 k	28·0 h + 17·2 k
16·1 h + 11·5 k	15·9 h + 21·3 k	17·2 h + 18·5 k	15·8 h + 24·8 k	18·6 h + 16·6 k	15·9 h + 25·6 k	18·6 h + 15·8 k	16·4 h + 29·4 k
10·7 h + 18·5 k	11·6 h + 26·6 k	12·8 h + 8·1 k	18·4 h + 9·1 k	17·5 h + 17·0 k	16·9 h + 24·0 k	14·1 h + 9·7 k	20·5 h + 14·6 k
26·3 h + 11·4 k	28·4 h + 19·2 k	10·6 h + 16·5 k	10·3 h + 21·5 k	16·8 h + 9·8 k	31·5 h + 1·6 k	26·6 h + 19·4 k	25·2 h + 32·2 k
20·7 h + 16·6 k	17·7 h + 30·0 k	24·3 h + 10·0 k	22·9 h + 16·9 k	19·5 h + 20·7 k	20·3 h + 25·8 k	26·0 h + 20·3 k	26·9 h + 31·1 k
9·5 h + 13·5 k	9·4 h + 23·3 k	9·3 h + 23·7 k	5·0 h + 33·5 k	20·3 h + 10·1 k	20·3 h + 15·9 k	1·5 h + 18·2 k	1·1 h + 30·8 k
18·2 h + 28·5 k	11·6 h + 44·6 k	29·9 h + 23·2 k	36·3 h + 22·2 k	26·6 h + 24·5 k	17·9 h + 40·6 k	10·4 h + 20·5 k	26·2 h + 15·5 k
19·1 h + 16·2 k	21·4 h + 23·5 k	20·8 h + 17·6 k	14·4 h + 29·1 k	11·8 h + 19·8 k	10·4 h + 28·1 k	16·5 h + 15·9 k	12·1 h + 30·6 k
14·0 h + 21·2 k	20·7 h + 25·5 k	13·7 h + 10·1 k	15·8 h + 13·6 k	15·6 h + 10·7 k	16·1 h + 18·2 k	14·6 h + 19·2 k	22·9 h + 23·8 k
22·9 h + 5·7 k	21·5 h + 17·2 k	20·7 h + 5·1 k	10·9 h + 20·1 k	3·8 h + 20·2 k	18·3 h + 12·6 k	24·1 h + 0·6 k	26·5 h + 9·1 k
21·2 h + 23·7 k	27·5 h + 26·4 k	16·3 h + 23·5 k	16·0 h + 28·3 k	19·1 h + 17·9 k	23·7 h + 20·3 k	14·8 h + 13·0 k	14·9 h + 23·1 k
18·5 h + 10·2 k	7·7 h + 31·6 k	11·8 h + 32·5 k	4·1 h + 45·8 k	3·4 h + 21·1 k	10·5 h + 20·4 k	4·9 h + 13·7 k	28·0 h + 1·8 k
5·2 h + 18·9 k	9·9 h + 24·2 k	3·4 h + 11·6 k	15·6 h + 3·6 k	10·2 h + 5·4 k	6·2 h + 16·0 k	1·4 h + 23·3 k	9·6 h + 26·5 k
17·38 h + 16·69 k	16·80 h + 27·02 k	15·78 h + 15·28 k	18·10 h + 18·43 k	16·14 h + 14·43 k	17·71 h + 19·69 k	17·47 h + 14·22 k	21·41 h + 21·73 k

Let the lengths of the spaces on the silver scale be

$$[0\cdot 2] = \frac{2\,\mathrm{I}}{10} + z$$

$$[0\cdot 3] = \frac{3\,\mathrm{I}}{10} + z'$$

and let $\alpha_1\ \alpha_2\ \alpha_3\ \alpha_4\ \alpha_5$ be the observed excesses of the space $[0\cdot 2]$ of the silver scale over $[a\cdot 2]$ $[2\cdot 4]$ $[4\cdot 6]$ $[6\cdot 8]$ $[8\cdot b]$ of **OF**, and $\beta_1\ \beta_2\ \beta_3\ \beta_4$ the excesses of $[0\cdot 3]$ silver scale over $[a\cdot 3]$ $[3\cdot 6]$ $[4\cdot 7]$ $[7\cdot b]$ of **OF**; then we have the following equations:—

$$\begin{aligned}
z - x_2 \qquad\quad - \alpha_1 &= 0 \qquad\qquad (11)\\
z - x_4 + x_2 - \alpha_2 &= 0\\
z - x_6 + x_4 - \alpha_3 &= 0\\
z - x_8 + x_6 - \alpha_4 &= 0\\
z \qquad\ + x_8 - \alpha_5 &= 0\\
z' \quad x_3 \qquad\quad - \beta_1 &= 0\\
z' - x_6 + x_3 - \beta_2 &= 0\\
z' - x_7 + x_4 - \beta_3 &= 0\\
z' \qquad\ + x_7 - \beta_4 &= 0
\end{aligned}$$

We have here 9 equations containing eight unknown quantities. Solving them by the method of least squares—

$$\begin{aligned}
0 &= 5z - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5\\
0 &= 4z' + x_4 - x_6 - \beta_1 - \beta_2 - \beta_3 - \beta_4\\
0 &= +2x_2 - x_4 + \alpha_1 - \alpha_2\\
0 &= +2x_3 - x_6 + \beta_1 - \beta_2\\
0 &= z' - x_2 + 3x_4 - x_6 - x_7 + \alpha_2 - \alpha_3 - \beta_3\\
0 &= -z' - x_3 - x_4 + 3x_6 - x_8 + \alpha_3 - \alpha_4 + \beta_2\\
0 &= -x_4 + 2x_7 + \beta_3 - \beta_4\\
0 &= -x_6 + 2x_8 + \alpha_4 - \alpha_5
\end{aligned}$$

Write A B C D E F G H for the absolute terms of these equations, and the values of $z\ z'\ x_2\ x_3\ x_4\ x_6\ x_7\ x_8$ will, by the solution of the equations, be found to be:

$$\begin{aligned}
0 &= 5z + A \qquad\qquad (12)\\
0 &= 20z' + 6B - C + D - 2E + 2F - G + H\\
0 &= 20x_2 - B + \tfrac{27}{2}C + \tfrac{3}{2}D + 7E + 3F + \tfrac{7}{2}G + \tfrac{3}{2}H\\
0 &= 20x_3 + B + \tfrac{3}{2}C + \tfrac{27}{2}D + 3E + 7F + \tfrac{3}{2}G + \tfrac{7}{2}H\\
0 &= 10x_4 - B + \tfrac{7}{2}C + \tfrac{3}{2}D + 7E + 3F + \tfrac{7}{2}G + \tfrac{3}{2}H\\
0 &= 10x_6 + B + \tfrac{3}{2}C + \tfrac{7}{2}D + 3E + 7F + \tfrac{3}{2}G + \tfrac{7}{2}H\\
0 &= 20x_7 - B + \tfrac{7}{2}C + \tfrac{3}{2}D + 7E + 3F + \tfrac{27}{2}G + \tfrac{3}{2}H\\
0 &= 20x_8 + B + \tfrac{3}{2}C + \tfrac{7}{2}D + 3E + 7F + \tfrac{3}{2}G + \tfrac{27}{2}H
\end{aligned}$$

Now, each of the absolute terms in the original equations is the mean of 25 different determinations; consequently the weights of x_2 x_3 x_4 x_6 x_7 x_8 are:

$$\begin{array}{lll} x_2 & \text{weight} & \frac{25 \cdot 40}{27} \\ x_3 & \text{,,} & \frac{25 \cdot 40}{27} \\ x_4 & \text{,,} & \frac{25 \cdot 10}{7} \\ x_6 & \text{,,} & \frac{25 \cdot 10}{7} \\ x_7 & \text{,,} & \frac{25 \cdot 40}{27} \\ x_8 & \text{,,} & \frac{25 \cdot 40}{27} \end{array} \qquad (13)$$

So that the 6 errors are determined with very nearly equal precision.

If we restore to A B C D E F G H their values in terms of α_1 α_2 α_3 α_4 α_5 β_1 β_2 β_3 β_4 we obtain values of x_2 x_3 x_4 x_6 x_7 x_8 as follows:

$$\begin{aligned} 40x_2 &= -27\alpha_1 + 13\alpha_2 + 8\alpha_3 + 3\alpha_4 + 3\alpha_5 - 5\beta_1 - 5\beta_2 + 5\beta_3 + 5\beta_4 \quad (14) \\ 40x_3 &= -3\alpha_1 - 3\alpha_2 - 8\alpha_3 + 7\alpha_4 + 7\alpha_5 - 25\beta_1 + 15\beta_2 + 5\beta_3 + 5\beta_4 \\ 40x_4 &= -14\alpha_1 - 14\alpha_2 + 16\alpha_3 + 6\alpha_4 + 6\alpha_5 - 10\beta_1 - 10\beta_2 + 10\beta_3 + 10\beta_4 \\ 40x_6 &= -6\alpha_1 - 6\alpha_2 - 16\alpha_3 + 14\alpha_4 + 14\alpha_5 - 10\beta_1 - 10\beta_2 + 10\beta_3 + 10\beta_4 \\ 40x_7 &= -7\alpha_1 - 7\alpha_2 + 8\alpha_3 + 3\alpha_4 + 3\alpha_5 - 5\beta_1 - 5\beta_2 - 15\beta_3 + 25\beta_4 \\ 40x_8 &= -3\alpha_1 - 3\alpha_2 - 8\alpha_3 - 13\alpha_4 + 27\alpha_5 - 5\beta_1 - 5\beta_2 + 5\beta_3 + 5\beta_4 \end{aligned}$$

The values of α_1 α_2 . . . . β_4 obtained from Tables V., VI. are:

$$\begin{aligned} \alpha_1 &= 0{\cdot}64\,h + 7{\cdot}58\,k \qquad (15) \\ \alpha_2 &= 1{\cdot}79\,h + 16{\cdot}74\,k \\ \alpha_3 &= 2{\cdot}43\,h + 5{\cdot}65\,k \\ \alpha_4 &= 1{\cdot}25\,h + 13{\cdot}20\,k \\ \alpha_5 &= 2{\cdot}26\,h + 12{\cdot}38\,k \\ \beta_1 &= -0{\cdot}58\,h + 10{\cdot}33\,k \\ \beta_2 &= 2{\cdot}32\,h + 3{\cdot}15\,k \\ \beta_3 &= 1{\cdot}41\,h + 5{\cdot}26\,k \\ \beta_4 &= 3{\cdot}94\,h + 7{\cdot}51\,k \end{aligned}$$

whence we get finally—

$$\begin{aligned} x_2 &= 1{\cdot}35\,h + 3{\cdot}28\,k \qquad (16) \\ x_3 &= 1{\cdot}85\,h - 2{\cdot}16\,k \\ x_4 &= 1{\cdot}55\,h - 2{\cdot}59\,k \\ x_6 &= 0{\cdot}79\,h + 2{\cdot}87\,k \\ x_7 &= 2{\cdot}04\,h - 0{\cdot}17\,k \\ x_8 &= 0{\cdot}90\,h + 1{\cdot}02\,k \end{aligned}$$

4.

Each of the tenths [2·3] [6·7] is subdivided into ten parts, and it remains to ascertain the errors of these hundredth-lines.

It is impossible to adjust a microscope to focus with the certainty of having no error in such adjustment. If a microscope be repeatedly brought to focus over a given surface, and if the distance of the object glass from the surface were measured each time, small differences would be found, very small indeed, but yet sufficient to influence the measure of a given space upon the surface under view. The error arising in a measurement from this cause is proportional to the length measured: hence it is not desirable to measure larger quantities than can be helped. The measurement of the hundredths in each tenth was effected in this manner: The scale is first levelled and brought to focus with the greatest nicety; the left-hand hundredth is brought into the field and placed accurately in the centre, so that the defining lines are equidistant from and on opposite sides of the zero position of the wires. The left line is bisected (with one micrometer reading), and then the cross is carried over to the right line, which is bisected, and the micrometer read. The scale (focus remaining unaltered) is now run $\frac{1}{100}$ of an inch to the left, until the second hundredth occupies exactly the position just vacated by the first. The cross will be found almost bisecting the right line; the bisection is made, the micrometer read, and the cross brought over by the revolution of the micrometer screw to the left line, which is bisected and read. The scale is again moved one hundredth of an inch to the left, and the third hundredth occupies the position just vacated by the second; it is then measured,—and so on. When the scale has been moved ten times one hundredth of an inch to the left, the last hundredth will be in the centre of the field of view, and its measurement closes the operation. We have now from 20 micrometer readings the measures of the ten spaces, each measure affected with the *same error, arising from the error of focus,* inasmuch as the focus remained unaltered during the operation. If this be called one series, each series will be affected with a *different* error as the focus is re-adjusted for each series.

The hundredths in each tenth have been measured twenty times each with each microscope in the manner explained above. The mean results of these measures, with their probable errors, are given in the following Table, expressed in divisions of the respective micrometers. The hundredths are numbered from left to right.

TABLE VII.

	Tenth [2·3]		Tenth [6·7]		
Number of Hundredths.	H	K	H	K	Number of Hundredths.
1	356·99 ± ·10	355·49 ± ·11	346·54 ± ·09	345·84 ± ·08	1
2	348·18 ± ·12	346·83 ± ·12	350·69 ± ·09	349·22 ± ·10	2
3	346·63 ± ·13	345·33 ± ·12	346·47 ± ·09	345·27 ± ·10	3
4	348·30 ± ·12	346·84 ± ·10	350·43 ± ·09	349·31 ± ·08	4
5	348·77 ± ·15	347·52 ± ·12	347·53 ± ·08	346·28 ± ·10	5
6	350·57 ± ·14	348·95 ± ·12	349·44 ± ·08	348·12 ± ·10	6
7	348·48 ± ·06	347·64 ± ·12	347·08 ± ·12	345·59 ± ·12	7
8	347·78 ± ·08	345·90 ± ·08	351·56 ± ·11	350·50 ± ·12	8
9	348·90 ± ·11	347·29 ± ·12	349·54 ± ·10	347·85 ± ·11	9
10	346·19 ± ·08	344·95 ± ·09	351·84 ± ·10	350·38 ± ·10	10

N.B.—The probable errors are determined from the differences of the 20 individual measures in each case with their mean.

From these measures we may readily find the errors of the nine subdividing lines in each tenth. If $n_1\ n_2 \cdots n_{10}$ be the successive measures of the tenths, the errors of division will be:

$$\begin{array}{l}
\tfrac{1}{10}\,(9\,n_1 - n_2 - n_3 - n_4 - n_5 - n_6 - n_7 - n_8 - n_9 - n_{10}) \qquad (17)\\
\tfrac{1}{10}\,(8\,n_1 + 8\,n_2 - 2\,n_3 - 2\,n_4 - 2\,n_5 - 2\,n_6 - 2\,n_7 - 2\,n_8 - 2\,n_9 - 2\,n_{10})\\
\tfrac{1}{10}\,(7\,n_1 + 7\,n_2 + 7\,n_3 - 3\,n_4 - 3\,n_5 - 3\,n_6 - 3\,n_7 - 3\,n_8 - 3\,n_9 - 3\,n_{10})\\
\tfrac{1}{10}\,(6\,n_1 + 6\,n_2 + 6\,n_3 + 6\,n_4 - 4\,n_5 - 4\,n_6 - 4\,n_7 - 4\,n_8 - 4\,n_9 - 4\,n_{10})\\
\tfrac{1}{10}\,(5\,n_1 + 5\,n_2 + 5\,n_3 + 5\,n_4 + 5\,n_5 - 5\,n_6 - 5\,n_7 - 5\,n_8 - 5\,n_9 - 5\,n_{10})\\
\tfrac{1}{10}\,(4\,n_1 + 4\,n_2 + 4\,n_3 + 4\,n_4 + 4\,n_5 + 4\,n_6 - 6\,n_7 - 6\,n_8 - 6\,n_9 - 6\,n_{10})\\
\tfrac{1}{10}\,(3\,n_1 + 3\,n_2 + 3\,n_3 + 3\,n_4 + 3\,n_5 + 3\,n_6 + 3\,n_7 - 7\,n_8 - 7\,n_9 - 7\,n_{10})\\
\tfrac{1}{10}\,(2\,n_1 + 2\,n_2 + 2\,n_3 + 2\,n_4 + 2\,n_5 + 2\,n_6 + 2\,n_7 + 2\,n_8 - 8\,n_9 - 8\,n_{10})\\
\tfrac{1}{10}\,(n_1 + n_2 + n_3 + n_4 + n_5 + n_6 + n_7 + n_8 + n_9 - 9\,n_{10})
\end{array}$$

If we substitute in these expressions the results shown in the preceding Table, we shall obtain the errors of division correctly; but it would be incorrect to infer the probable errors of these quantities in the usual way from the probable errors shown in the Table when introduced into the formulæ (16). The probable errors in the preceding Table VI. are affected, and necessarily, by the errors of focal adjustment in the microscope; whereas if we apply the formulæ (16) to each *series* of micrometer measures as described above, we shall eliminate the focal error from that series; or, in other words, the errors of division resulting from any single series of measures are not affected by the focal error of that series. This is evident, because the sum of the coefficients of the quantities n is *zero* for each error of division; consequently a constant error disappears. Therefore, we must compute the errors of division from each series, and then compare individual results with their means to obtain the real probable errors. The following Table contains the errors of division with probable errors of such determinations.

TABLE VIII.

Errors of Subdivisions of [2·3]			Errors of Subdivisions of [6·7]		
Line.	H	K	H	K	Line.
1	+ 7·91 ± ·10	+ 7·82 ± ·11	− 2·57 ± ·09	− 2·00 ± ·07	1
2	+ 7·01 ± ·14	+ 6·98 ± ·10	− 0·99 ± ·16	− 0·61 ± ·08	2
3	+ 4·56 ± ·16	+ 4·64 ± ·16	− 3·63 ± ·16	− 3·17 ± ·11	3
4	+ 3·78 ± ·10	+ 3·81 ± ·13	− 2·31 ± ·15	− 1·70 ± ·13	4
5	+ 3·47 ± ·17	+ 3·66 ± ·19	− 3·89 ± ·11	− 3·25 ± ·13	5
6	+ 4·96 ± ·18	+ 4·94 ± ·17	− 3·56 ± ·16	− 2·97 ± ·18	6
7	+ 4·36 ± ·15	+ 4·91 ± ·15	− 5·59 ± ·18	− 5·21 ± ·13	7
8	+ 3·06 ± ·13	+ 3·14 ± ·12	− 3·14 ± ·13	− 2·55 ± ·13	8
9	+ 2·88 ± ·08	+ 2·76 ± ·06	− 2·72 ± ·09	− 2·54 ± ·10	9

These results are expressed in terms of the divisions of the respective microscopes. They are not the *absolute* errors of division of the lines, but are only the errors *as subdividing* into tenths the spaces [2·3] [6·7].

There is one line in particular in each tenth which demands especial attention; namely, in [2·3], the sixth line from 2 towards 3, and fourth from 3 towards 2: we shall designate this line by the letter τ; and in [6·7], the second line from 6 towards 7, and eighth from 7 towards 6: this line we shall designate by the letter μ. These two lines are those which come into play in the measurement of the Toise and Metre respectively; we shall therefore give the actual results of the micrometer measurements in the twenty series. They are shown in the following Table:—

TABLE IX.

Tenth [2·3]				Tenth [6·7]			
H		K		H		K	
[2·τ]	[τ·3]	[2·τ]	[τ·3]	[6·μ]	[μ·7]	[6·μ]	[μ·7]
2099·6	1392·0	2093·2	1384·0	698·4	2794·2	694·8	2780·1
2097·8	1392·0	2088·7	1384·6	698·3	2796·0	694·2	2782·0
2098·2	1388·8	2089·8	1383·6	696·9	2791·6	694·5	2784·3
2097·7	1392·7	2090·9	1388·4	698·0	2794·4	694·5	2783·2
2100·3	1391·5	2087·3	1385·3	698·0	2791·7	694·8	2779·6
2102·8	1391·2	2094·3	1387·4	697·3	2792·4	695·0	2784·0
2096·1	1391·2	2090·9	1383·0	696·3	2795·0	693·9	2784·1
2104·7	1389·8	2091·3	1384·9	697·7	2793·9	695·7	2782·8
2099·9	1392·0	2094·7	1388·2	696·2	2791·4	694·0	2781·0
2102·9	1392·5	2089·9	1387·7	696·0	2795·1	697·0	2785·3
2099·4	1393·7	2085·1	1383·2	697·4	2793·7	695·4	2786·5
2097·7	1388·7	2092·5	1387·1	698·1	2793·4	694·4	2781·7
2095·9	1390·4	2087·0	1384·6	697·5	2794·1	694·7	2780·9
2097·7	1389·9	2090·9	1385·6	697·9	2794·9	696·2	2788·7
2101·6	1391·6	2092·3	1386·7	696·3	2793·0	695·1	2780·7
2102·7	1392·5	2094·6	1385·5	696·6	2800·1	696·2	2787·6
2096·1	1390·2	2089·9	1384·0	696·6	2792·4	694·4	2782·9
2099·0	1392·5	2093·5	1390·9	697·4	2792·3	695·4	2781·9
2095·2	1391·6	2089·9	1384·7	695·4	2795·1	696·3	2787·0
2103·1	1391·9	2092·1	1386·0	698·3	2793·0	694·5	2781·2

Expressed in Micrometer Divisions of respective Microscopes.

The first and second columns, and the third and fourth, give the values of $n_1 + \dots n_6$ and $n_7 + \dots n_{10}$. The fifth and sixth columns, and the seventh and eighth, give $n_1 + n_2$ and $n_3 + \dots n_{10}$. By (17) the errors of division of the lines τ and μ are

$$\tau \dots\dots \tfrac{4}{10}(n_1 + \dots n_6) - \tfrac{6}{10}(n_7 + \dots n_{10}) \qquad (18)$$
$$\mu \dots\dots \tfrac{8}{10}(n_1 + n_2) - \tfrac{2}{10}(n_3 + \dots n_{10})$$

Applying these formulæ to the quantities in the preceding Table, we get the following results for the errors of division:—

TABLE X.

Error of τ		Error of μ	
H	K	H	K
+ 4·64	+ 6·88	− 0·12	− 0·18
3·92	4·72	− 0·56	− 1·04
6·00	5·76	− 0·80	− 1·26
3·46	3·32	− 0·48	− 1·04
5·22	3·74	+ 0·06	− 0·08
6·40	5·28	− 0·64	− 0·80
3·72	6·56	− 1·96	− 1·70
8·00	5·58	− 0·62	− 0·00
4·76	4·96	− 1·32	− 1·00
5·66	3·34	− 2·22	+ 0·54
3·54	4·12	− 0·82	− 0·98
5·86	4·74	− 0·20	− 0·82
4·12	4·04	− 0·82	− 0·42
5·14	5·00	− 0·66	− 0·78
5·68	4·90	− 1·56	− 0·06
5·58	6·54	− 2·74	− 0·56
4·32	5·56	− 1·20	− 1·06
4·10	2·86	− 0·54	− 0·06
3·12	5·14	− 2·70	− 0·36
+ 6·10	+ 5·24	+ 0·04	− 0·64

We shall consider the probable errors of these determinations after the values of h and k are ascertained.

5.

In Table IX., by adding together the quantities $[2\cdot\tau] + [\tau\cdot 3]$ and $[6\cdot\mu] + [\mu\cdot 7]$, we get the number of divisions of H or K which go to the spaces $[2\cdot 3]$ $[6\cdot 7]$. Now, from (3)

$$[2\cdot 3] = \frac{1}{10} + x_3 - x_2 + \frac{x_b}{10} + \frac{x_g}{60} = \frac{1}{10} + 0{\cdot}49\,h - 5{\cdot}40\,k \qquad (19)$$

$$[6\cdot 7] = \frac{1}{10} + x_7 - x_6 + \frac{x_b}{10} + \frac{x_g}{60} = \frac{1}{10} + 1{\cdot}23\,h - 3{\cdot}01\,k \qquad (20)$$

We must here anticipate our results so far as to remark that $h - k$ is less than one 250th part of h or k. We may, therefore, in the above expression put h for k or k for h, so that $[2\cdot 3] = \frac{1}{10}\,\mathrm{I} - 4{\cdot}91\,h$ or $\frac{1}{10}\,\mathrm{I} - 4{\cdot}91\,k$; and $[6\cdot 7] = \frac{1}{10}\,\mathrm{I} - 1{\cdot}78\,h$ or $\frac{1}{10}\,\mathrm{I} - 1{\cdot}78\,k$. In order, therefore, to ascertain the true number of divisions of H or K which are equivalent to a tenth of an inch, we must increase every measure of $[2\cdot 3]$ by + 4·91 divisions, and every measure of $[6\cdot 7]$ by + 1·78 divisions.

The quantities taken from Table IX., so modified, will stand thus :—

TABLE XI.

H		K	
No. of Divisions to $\frac{1}{10}$ I.	Error, or excess of individual results above their mean.	No. of Divisions to $\frac{1}{10}$ I.	Error, or excess of individual results above their mean.
3496·5	+ 2·2	3482·1	+ 1·2
3494·7	+ 0·4	3478·2	− 2·7
3491·9	− 2·4	3478·3	− 2·6
3495·3	+ 1·0	3484·2	+ 3·3
3496·7	+ 2·4	3477·5	− 3·4
3498·9	+ 4·6	3486·6	+ 5·7
3492·2	− 2·1	3478·8	− 2·1
3499·4	+ 5·1	3481·1	+ 0·2
3496·8	+ 2·5	3487·8	+ 6·9
3500·3	+ 6·0	3482·5	+ 1·6
3498·0	+ 3·7	3473·2	− 7·7
3491·3	− 3·0	3484·5	+ 3·6
3491·2	− 3·1	3476·5	− 4·4
3492·5	− 1·8	3481·4	+ 0·5
3498·1	+ 3·8	3483·9	+ 3·0
3500·1	+ 5·8	3485·0	+ 4·1
3491·2	− 3·1	3478·8	− 2·1
3496·4	+ 2·1	3489·3	+ 8·4
3491·7	− 2·6	3479·5	− 1·4
3499·9	+ 5·6	3483·0	+ 2·1
3494·4	+ 0·1	3476·7	− 4·2
3496·1	+ 1·8	3478·0	− 2·9
3490·3	− 4·0	3480·6	− 0·3
3494·2	− 0·1	3479·5	− 1·4
3491·5	− 2·8	3476·2	− 4·7
3491·5	− 2·8	3480·8	− 0·1
3493·1	− 1·2	3479·8	− 1·1
3493·4	− 0·9	3480·3	− 0·6
3489·4	− 4·9	3476·8	− 4·1
3492·9	− 1·4	3484·1	+ 3·2
3492·9	− 1·4	3483·7	+ 2·8
3493·3	− 1·0	3477·9	− 3·0
3493·4	− 0·9	3477·4	− 3·5
3494·6	+ 0·3	3486·7	+ 5·8
3491·1	− 3·2	3477·6	− 3·3
3498·5	+ 4·2	3485·6	+ 4·7
3490·8	− 3·5	3479·1	− 1·8
3491·5	− 2·8	3479·1	− 1·8
3492·3	− 2·0	3485·1	+ 4·2
3493·1	− 1·2	3477·5	− 3·4

The sum of the squares of the errors in the measures by H is 367·54, in that by K 534·27. Consequently, the mean error in a single measurement of one tenth of an inch (in ten consecutive hundredths) by H is

$$\sqrt{\frac{367 \cdot 54}{40 - 1}} = \pm 3 \cdot 07$$

And the mean error in the same operation by K is

$$\sqrt{\frac{534 \cdot 27}{40 - 1}} = \pm 3 \cdot 70$$

The corresponding probable errors are

H	K
$\pm$ 2·07	$\pm$ 2·49

Now, each of the errors in the above Table results from the sum of the errors of twenty single micrometer readings and the error due to a slight error in the focussing of the microscope. Let the probable errors resulting from this last cause be

$$\epsilon_H \text{ for H, and } \epsilon_K \text{ for K,}$$

on a measurement of 1000 divisions: then upon the tenth of an inch they will be

$$3 \cdot 494\, \epsilon_H \text{ and } 3 \cdot 481\, \epsilon_K$$

By a special investigation, the *probable error of a single micrometer reading* was found to be—

Observer, Captain Clarke, R.E. $\pm$ 0·32 micr. div.
" Quartermaster Steel, R.E. $\pm$ 0·31 "

the same result being obtained for either microscope.

Now the sum of the squares of 20 such errors is 2·00; consequently the probable error in the measure of one tenth of an inch by H or K will be

$$H \ldots\ldots \sqrt{2 \cdot 00 + 3 \cdot 494^2 \cdot \epsilon^2_H}$$
$$K \ldots\ldots \sqrt{2 \cdot 00 + 3 \cdot 481^2 \cdot \epsilon^2_K}$$

If now we compare this with the numerical quantities actually obtained, it appears that

$$2 \cdot 00 + 3 \cdot 494^2\, \epsilon^2_H = 4 \cdot 28$$
$$2 \cdot 00 + 3 \cdot 481^2\, \epsilon^2_K = 6 \cdot 20$$

$$\therefore\ \epsilon_H = \frac{\sqrt{2 \cdot 28}}{3 \cdot 494} = \pm 0 \cdot 432 \text{ micr. div.} \qquad (21)$$

$$\epsilon_K = \frac{\sqrt{4 \cdot 20}}{3 \cdot 481} = \pm 0 \cdot 589 \text{ micr. div.} \qquad (22)$$

Consequently, if a space of n thousand divisions be measured by H or K, the probable error of such measure will be

$$H \ldots\ldots \pm \sqrt{0 \cdot 432^2\, n^2 + \cdot 20} = \sqrt{\cdot 187\, n^2 + \cdot 20} \qquad (23)$$

$$K \ldots\ldots \pm \sqrt{0 \cdot 589^2\, n^2 + \cdot 20} = \sqrt{\cdot 347\, n^2 + \cdot 20} \qquad (24)$$

If, for example, 250 divisions be measured, the probable error of the single measurement, expressed in divisions, is

H . . . . . $\pm$ 0·46
K . . . . . $\pm$ 0·49

If there were no focal error these quantities would be simply 0·45. We gather, then, this very important result, that even in measuring such an unusually large space as 250 micrometer divisions, the probable error of a single determination is only increased from $\pm$ 0·45 to $\pm$ 0·46 or $\pm$ 0·49 by the probable error in focal adjustment of the microscope.

It appears from our last Table that the number of divisions to one tenth of an inch in the two microscopes is

$$\text{for H} \ldots\ldots 3494{\cdot}3 \pm \frac{2{\cdot}07}{\sqrt{40}}$$

$$\text{,, K} \ldots\ldots 3480{\cdot}9 \pm \frac{2{\cdot}49}{\sqrt{40}}$$

Consequently, the number of divisions to *one inch* is

$$\text{for H} \ldots\ldots 34943 \pm 3{\cdot}3$$
$$\text{,, K} \ldots\ldots 34809 \pm 3{\cdot}9$$

Therefore h, k, being the values of *one division* in H and K respectively, and 1 an inch,

$$h = \frac{1}{34943 \pm 3{\cdot}3} \qquad (25)$$

$$k = \frac{1}{34809 \pm 3{\cdot}9} \qquad (26)$$

6.

It is convenient to have a unit of reference for small quantities, as micrometer measures, expansions, &c. Such quantities, when expressed in inches, are encumbered with an inconvenient number of decimals, and they cannot well be expressed in micrometer divisions, as no two microscopes have divisions equivalent in magnitude. If we express the values of a division of H and K first obtained in inches, they become

$$h = 0{\cdot}0000286180 \pm {\cdot}0000000027$$
$$k = 0{\cdot}0000287282 \pm {\cdot}0000000032$$

The yard being the unit of length in this country, it seems reasonable to take as the unit of reference for small quantities the *millionth part of the yard*. Expressed in this unit, the quantities above become

$$h = 0{\cdot}79494 \pm {\cdot}00008 \qquad (27)$$
$$k = 0{\cdot}79800 \pm {\cdot}00009 \qquad (28)$$

So that a division of either micrometer is about $\frac{8}{10}$ of the unit of reference.

In future all small quantities will be expressed with reference to this unit, the millionth part (·000001) of the yard, unless expressly stated otherwise.

7.

If we write out in full the set of 60 equations from which x_g is obtained in equation (4), and of which the quantities given in Table I. form the absolute terms, and if we substitute the actual values of h and k in these equations, we get the sum of the squares of the errors 5·26. Now in the 60 equations there are 30 quantities $z_1 \ldots\ldots z_{30}$, and one quantity x, in all 31 unknown quantities: consequently the probable error of a single equation is

$$\pm\, 0{\cdot}674 \sqrt{\frac{5{\cdot}26}{60 - 31}} = \pm\, 0{\cdot}287$$

and therefore for the value of x_g we have

$$x_g = -0{\cdot}87 \pm \frac{{\cdot}287}{\sqrt{60}} = -0{\cdot}87 \pm {\cdot}037 \qquad (29)$$

The equations (5) are in number 209: those in z'' are to be multiplied by $\sqrt{\frac{25}{17}}$ in forming the errors. If we substitute in these equations and in (8), (9), and (10) the values of h and k, we get the sum of the squares of the 209 errors = 16·33. Now the number of unknown quantities in these equations is 72, consequently the probable error of a single equation is

$$\pm 0{\cdot}674 \sqrt{\frac{16{\cdot}33}{209 - 72}} = \pm 0{\cdot}233 \qquad (30)$$

Consequently by (7) the probable errors of x_b x_c x_d x_e x_f are

$$\pm \frac{{\cdot}233}{5}\sqrt{\frac{11}{18}} = \pm {\cdot}036 \qquad (31)$$

$$\pm \frac{{\cdot}233}{5}\sqrt{\frac{17}{18}} = \pm {\cdot}045$$

$$\pm \frac{{\cdot}233}{5}\sqrt{\frac{9}{18}} = \pm {\cdot}033$$

$$\pm \frac{{\cdot}233}{5}\sqrt{\frac{17}{18}} = \pm {\cdot}045$$

$$\pm \frac{{\cdot}233}{5}\sqrt{\frac{11}{18}} = \pm {\cdot}036$$

The equations (11) are in reality 225 in number, each of those written down being the mean of 25 similar equations. On substituting the values of h and k in (11), (15), and (16), we obtain for the sum of the squares of the 225 errors 71·08. The number of unknown quantities is 8, consequently the probable error of an equation is

$$\pm 0{\cdot}674 \sqrt{\frac{71{\cdot}08}{225 - 8}} = \pm 0{\cdot}386 \qquad (32)$$

Therefore the probable errors of x_2 x_3 x_4 x_6 x_7 x_8 are—see (13),

$$\pm \frac{{\cdot}386}{5}\sqrt{\frac{27}{40}} = \pm 0{\cdot}063 \qquad (33)$$

$$\pm \frac{{\cdot}386}{5}\sqrt{\frac{27}{40}} = \pm 0{\cdot}063$$

$$\pm \frac{{\cdot}386}{5}\sqrt{\frac{7}{10}} = \pm 0{\cdot}064$$

$$\pm \frac{{\cdot}386}{5}\sqrt{\frac{7}{10}} = \pm 0{\cdot}064$$

$$\pm \frac{{\cdot}386}{5}\sqrt{\frac{27}{40}} = \pm 0{\cdot}063$$

$$\pm \frac{{\cdot}386}{5}\sqrt{\frac{27}{40}} = \pm 0{\cdot}063$$

We now proceed to consider the errors of the lines τ and μ. The quantities in Table X. being multiplied by the values of h and k become as in the next Table.

x_τ	Errors.	x_μ	Errors.
+ 3·69	− 0·25	− 0·10	+ 0·54
3·12	− 0·82	− 0·44	+ 0·20
4·77	+ 0·83	− 0·64	+ 0·00
2·75	− 1·18	− 0·38	+ 0·26
4·15	+ 0·22	+ 0·05	+ 0·69
5·09	+ 1·15	− 0·51	+ 0·13
2·96	− 0·98	− 1·56	− 0·92
6·36	+ 2·42	− 0·49	+ 0·15
3·78	− 0·15	− 1·05	− 0·41
4·50	+ 0·56	− 1·76	− 1·12
2·81	− 1·12	− 0·65	− 0·01
4·66	+ 0·72	− 0·16	+ 0·48
3·27	− 0·66	− 0·65	− 0·01
4·09	+ 0·15	− 0·52	+ 0·12
4·52	+ 0·58	− 1·24	− 0·60
4·44	+ 0·50	− 2·18	− 1·54
3·43	− 0·50	− 0·95	− 0·31
3·26	− 0·67	− 0·43	+ 0·21
2·48	− 1·45	− 2·15	− 1·51
4·85	+ 0·91	+ 0·03	+ 0·67
5·49	+ 1·55	− 0·14	+ 0·50
3·77	− 0·17	− 0·83	− 0·19
4·60	+ 0·66	− 1·00	− 0·36
2·65	− 1·29	− 0·83	− 0·19
2·98	− 0·95	− 0·06	+ 0·58
4·21	+ 0·28	− 0·64	+ 0·00
5·23	+ 1·30	− 1·36	− 0·72
4·45	+ 0·52	− 0·00	+ 0·64
3·96	+ 0·02	− 0·80	− 0·16
2·66	− 1·27	+ 0·43	+ 1·07
3·29	− 0·65	− 0·78	− 0·14
3·78	− 0·15	− 0·65	− 0·01
3·22	− 0·71	− 0·33	+ 0·31
3·99	+ 0·05	− 0·62	+ 0·02
3·91	− 0·02	− 0·05	+ 0·59
5·22	+ 1·28	− 0·45	+ 0·19
4·44	+ 0·50	− 0·84	− 0·20
2·28	− 1·65	− 0·05	+ 0·59
4·10	+ 0·17	− 0·29	+ 0·35
+ 4·18	+ 0·25	− 0·51	+ 0·13

The sum of the squares of the errors of x_τ is 32·635, consequently the probable error of the determination is

$$\pm \cdot 674 \sqrt{\frac{32 \cdot 635}{39 \times 40}} = \pm \cdot 098 \qquad (34)$$

The sum of the squares of the errors for x_μ is 12·940, consequently the probable error of the determination is

$$\pm \cdot 674 \sqrt{\frac{12 \cdot 940}{39 \times 40}} = \pm \cdot 061 \qquad (35)$$

The arithmetical means of the quantities in the columns headed x_τ and x_μ are + 3·93 and − 0·64. These are not the absolute errors of division of the lines τ and μ, but only with reference to the lines 2, 3 and 6, 7.

The definite results, then, at which we have arrived are:

$$\begin{aligned} x_2 &= +\ 3{\cdot}69 \pm {\cdot}063 \\ x_\tau &= +\ 3{\cdot}93 \pm {\cdot}098 \\ x_3 &= -\ 0{\cdot}25 \pm {\cdot}063 \\ x_4 &= -\ 0{\cdot}84 \pm {\cdot}064 \\ x_6 &= +\ 2{\cdot}92 \pm {\cdot}064 \\ x_\mu &= -\ 0{\cdot}64 \pm {\cdot}061 \\ x_7 &= +\ 1{\cdot}49 \pm {\cdot}063 \\ x_8 &= +\ 1{\cdot}53 \pm {\cdot}063 \\ x_b &= +\ 0{\cdot}29 \pm {\cdot}036 \\ x_c &= +\ 2{\cdot}03 \pm {\cdot}045 \\ x_d &= +\ 1{\cdot}75 \pm {\cdot}033 \\ x_e &= +\ 1{\cdot}43 \pm {\cdot}045 \\ x_f &= +\ 2{\cdot}93 \pm {\cdot}036 \\ x_g &= -\ 0{\cdot}87 \pm {\cdot}037 \end{aligned} \qquad (36)$$

And from these it remains to determine the absolute errors of the lines and the probable errors of such determinations. In the first place,

$$\begin{aligned} [2 \cdot \tau] &= \tfrac{6}{10}\,[2 \cdot 3] + x_\tau \\ [6 \cdot \mu] &= \tfrac{2}{10}\,[6 \cdot 7] + x_\mu \end{aligned} \qquad (37)$$

And by (3),

$$[2 \cdot 3] = \frac{I}{10} + x_3 - x_2 + \frac{x_b}{10} + \frac{x_g}{60}$$

$$[6 \cdot 7] = \frac{I}{10} + x_7 - x_6 + \frac{x_b}{10} + \frac{x_g}{60}$$

$$\therefore\ [2 \cdot \tau] = \frac{6I}{100} - \tfrac{6}{10}x_2 + x_\tau + \tfrac{6}{10}x_3 + \frac{6\,x_b}{100} + \frac{x_g}{100}$$

$$[6 \cdot \mu] = \frac{2I}{100} - \tfrac{2}{10}x_6 + x_\mu + \tfrac{2}{10}x_7 + \frac{2\,x_b}{100} + \frac{x_g}{300}$$

$$\left.\begin{aligned} \therefore\ [a \cdot \tau] &= \frac{26I}{100} + \tfrac{4}{10}x_2 + x_\tau + \tfrac{6}{10}x_3 + \frac{26\,x_b}{100} + \frac{26\,x_g}{600} \\ [a \cdot \mu] &= \frac{62I}{100} + \tfrac{8}{10}x_6 + x_\mu + \tfrac{2}{10}x_7 + \frac{62\,x_b}{100} + \frac{62\,x_g}{600} \end{aligned}\right\} \qquad (38)$$

With respect to the probable errors of the determination of errors of division of the lines 2, 3, 4, 6, 7, 8, it will be seen from (3) that each one depends on the probable errors of three other quantities independently determined; for instance, if E_n be the probable error of the determination of x_n the probable error of $[a \cdot 2]$ is

$$= \sqrt{(E_2)^2 + (\tfrac{2}{10}\,E_b)^2 + (\tfrac{2}{60}\,E_g)^2}$$

but in the case of $[a \cdot \tau]$ and $[a \cdot \mu]$ there is a difference, for the former involves

$$\frac{4\,x_2 + 6\,x_3}{10}$$

but x_2 and x_3 are not independently determined. From equations (12) it appears that the weight of the determination of $\alpha x_2 + \beta x_3$ is

$$\frac{40}{27\alpha^2 + 6\alpha\beta + 27\beta^2}$$

consequently the reciprocal of the weight of $\frac{4}{10}x_2 + \frac{6}{10}x_3$ is

$$\frac{1548}{4000}$$

and the corresponding probable error

$$\pm\frac{\cdot 386}{5}\sqrt{\frac{1548}{4000}} = \pm\,{\cdot}048 \qquad (39)$$

and $[a\cdot\mu]$ involves

$$\tfrac{8}{10}x_6 + \tfrac{2}{10}x_7$$

By equations (12) the weight of the determination of $\alpha x_6 + \beta x_7$ is

$$\frac{40}{28\alpha^2 + 12\alpha\beta + 27\beta^2}$$

consequently the reciprocal of the weight of $\frac{8}{10}x_6 + \frac{2}{10}x_7$ is

$$\frac{2092}{4000}$$

and the corresponding probable error

$$\pm\frac{\cdot 386}{5}\sqrt{\frac{2092}{4000}} = \pm\,{\cdot}056 \qquad (40)$$

so that the probable errors of the determinations of $[a\cdot\tau]$ and $[a\cdot\mu]$ are respectively

$$\sqrt{({\cdot}048)^2 + (E_\tau)^2 + \left(\frac{26}{100}E_b\right)^2 + \left(\frac{26}{600}E_y\right)^2}$$

$$\sqrt{({\cdot}056)^2 + (E_\mu)^2 + \left(\frac{62}{100}E_b\right)^2 + \left(\frac{62}{600}E_y\right)^2}$$

Proceeding in this manner, we get the following final results :

$$[a\cdot 2] = \qquad + 3{\cdot}71 \pm {\cdot}063 \qquad (41)$$

$$[a\cdot\tau] = \frac{26}{100}\,\mathrm{I} + 5{\cdot}30 \pm {\cdot}100$$

$$[a\cdot 3] = \frac{3}{10}\,\mathrm{I} - 0{\cdot}20 \pm {\cdot}064$$

$$[a\cdot 4] = \frac{4}{10}\,\mathrm{I} - 0{\cdot}77 \pm {\cdot}066$$

$$[a\cdot 6] = \frac{6}{10}\,\mathrm{I} + 3{\cdot}00 \pm {\cdot}068$$

$$[a\cdot\mu] = \frac{62}{100}\,\mathrm{I} + 2{\cdot}08 \pm {\cdot}086$$

$$[a\cdot 7] = \frac{7}{10}\,\mathrm{I} + 1{\cdot}58 \pm {\cdot}068$$

$$[a\cdot 8] = \frac{8}{10}\,\mathrm{I} + 1{\cdot}65 \pm {\cdot}070$$

$$[a \cdot b] = \text{I} + 0{\cdot}14 \pm {\cdot}037$$
$$[a \cdot c] = 2\,\text{I} + 1{\cdot}74 \pm {\cdot}047$$
$$[a \cdot d] = 3\,\text{I} + 1{\cdot}31 \pm {\cdot}038$$
$$[a \cdot e] = 4\,\text{I} + 0{\cdot}84 \pm {\cdot}051$$
$$[a \cdot f] = 5\,\text{I} + 2{\cdot}20 \pm {\cdot}047$$
$$[a \cdot g] = 6\,\text{I} - 0{\cdot}87 \pm {\cdot}037$$

8.

The absolute terms of equations (3) are the quantities given in Tables III., IV., and those in Table II. multiplied by $\sqrt{\frac{3}{2}}$ or rather $\sqrt{\frac{25}{17}}$. Now each quantity in Tables III., IV., is of the form; ($a_1 \ldots\ldots a_6\ b_1 \ldots\ldots b_6$ representing single micrometer readings)—

$$\frac{a_1 + a_2 + \ldots a_6}{6} h + \frac{b_1 + b_2 + \ldots b_6}{6} k$$

and the probable error of a single reading of H or K is $\pm {\cdot}31 \times {\cdot}8 = \pm {\cdot}25$; consequently the probable error of any of the quantities in III. or IV., so far as mere errors of reading or observation are concerned, is

$$\pm {\cdot}25 \sqrt{\frac{12}{36}} = \pm \frac{{\cdot}25}{\sqrt{3}} = \pm 0{\cdot}14$$

Each of the quantities in Table II. is of the form

$$\frac{a_1 \ldots\ldots a_9}{9} h + \frac{b_1 \ldots\ldots b_9}{9} k$$

and its probable error

$$\pm {\cdot}25 \sqrt{\frac{18}{81}} = \pm \frac{{\cdot}25}{3} \sqrt{2}$$

which when multiplied by $\sqrt{\frac{3}{2}}$ becomes the same as the probable error of the quantities in Tables III., IV.

But we have seen (30) that the probable error of a single equation is $\pm {\cdot}233$, as determined by the solution of the equations. It appears, therefore, that the probable error of our results is nearly double what might be expected from errors of observation only. There are, therefore, other sources of error at work besides the mere defects of vision. What these may be it would be difficult to say, but one of them at any rate is known; *personal error*.

If any two observers bisect a given line under a microscope and the operation be repeated, microscope and object remaining steady, it may be found that in the long run there is a perceptible and systematic difference between their readings. This difference will in general be a very small quantity, but sometimes it is not so small as not to be detrimental to the results. If the quantity were absolutely constant for all lines, the error would entirely disappear in every comparison, but it appears that the difference or personal error varies slightly with different lines. This, of course, will directly affect results. The observations contained in Tables I., II., III., IV., were made altogether by two observers, Captain Clarke, R.E., and Quartermaster J. Steel, R.E., who differ systematically in the *bisection of a line* by from 1·5 to nearly 2 divisions. The greatest discrepancy resulting from personal error is to be found in the comparison of the two six-inch portions of **OF** or in the determination of the quantity x_y. The absolute terms of the equations (60 in number) from which x_g results are the 60 quantities in Table I. The probable error of each of these quantities due to errors of reading only is $\pm \frac{\cdot 25}{3} \sqrt{2} = \pm$ 0·118, as we have seen above. But by the errors exhibited after the solution of the equations we have found for the probable error of an equation $\pm$ 0·287, which is more than double the error resulting from readings only. If in this series the errors pertaining to the two observers be arranged separately, it is found that the *arithmetical mean* of the errors of one observer is + 0·14 from 32 observations, and that of the other is − 0·16 from 28 observations: showing an effect of personal error between the two of 0·30. This, however, is the *largest* difference that has been noticed. Personal error does not easily admit of being eliminated, unless there be a large number of observers engaged at the same work.

In equations (11) the absolute term is the difference of two quantities in Tables V., VI., it is, therefore, of the form

$$\frac{a_1 + a_2 + a_3 - a'_1 - a'_2 - a'_3}{3} h + \frac{b_1 + b_2 + b_3 - b'_1 - b'_2 - b'_3}{3} k$$

where a_1 b_1 .... are single micrometer readings. The probable error of this expression is

$$\pm \cdot 25 \sqrt{\frac{12}{9}} = \pm \frac{\cdot 50}{\sqrt{3}} = \pm \cdot 288$$

But after the resolution of the equations we found (32) the probable error of a single equation $\pm$ ·386. Here, again, we see that the error is larger than mere errors of observation will account for.

There is no other known source of error in this series of observations than errors of observation (readings or bisections) and personal error. Sudden changes of temperature there are none: the presence of the observer may raise the temperature of the bar perhaps 5 hundredths of a degree Fahrenheit during a visit, but this in such small lengths as are now under consideration will not be nearly sufficient to explain the residual errors.

9.

The portions of this scale which are referred to in the determinations of the length of the Toise and Metre are, for the Toise, that which lies between the line designated τ and that designated f, or $[\tau \cdot f]$; and for the Metre, that which lies between the lines μ and e, or $[\mu \cdot e]$. Now, from (2) and (38),

$$[\tau \cdot f] = \frac{474}{100} \text{I} - \tfrac{4}{10} x_2 - x_\tau - \tfrac{6}{10} x_3 - \tfrac{26}{100} x_b + x_f + \tfrac{474}{600} x_g \qquad (42)$$

$$[\mu \cdot e] = \frac{338}{100} \text{I} - \tfrac{8}{10} x_6 - x_\mu - \tfrac{2}{10} x_7 - \tfrac{62}{100} x_b + x_e + \tfrac{338}{600} x_g \qquad (43)$$

each involving the determinations of six errors of divisions.

With respect to the probable errors of $[\tau \cdot f]$ and $[\mu \cdot e]$ we see that they are each the combination of four partial probable errors; viz.,

	$[\tau \cdot f]$		$[\mu \cdot e]$
1	$\frac{4}{10} x_2 + \frac{6}{10} x_3$	1	$\frac{8}{10} x_6 + \frac{2}{10} x_7$
2	x_τ	2	x_μ
3	$-\frac{26}{100} x_b + x_f$	3	$-\frac{62}{100} x_b + x_e$
4	$\frac{474}{600} x_g$	4	$\frac{338}{600} x_g$

The probable error of the first quantity in each of these groups we have already considered. From equations (6) we see that the weight of the determination of $\alpha x_b + \beta x_f$ is

$$\frac{18}{11\,\alpha^2 + 2\,\alpha\beta + 11\,\beta^2}$$

If $\alpha = -\frac{26}{100}$; $\beta = 1$, this becomes

$$\frac{180000}{112236}$$

and consequently the probable error of the determination of $-\frac{26}{100}\,x_b + x_f$ is by (30)

$$\pm\,\frac{\cdot 233}{5}\sqrt{\frac{1122}{1800}} = \pm\,\cdot 037 \tag{44}$$

Again, by equations (6) the weight of the determination of $\alpha x_b + \beta x_o$ is

$$\frac{18}{11\,\alpha^2 + 4\,\alpha\beta + 17\,\beta^2}$$

Making $\alpha = -\frac{62}{100}$; $\beta = 1$, we have for the weight of the determination of $-\frac{62}{100}\,x_b + x_o$

$$\frac{180000}{187484}$$

Consequently the probable error is

$$\pm\,\frac{\cdot 233}{5}\sqrt{\frac{1875}{1800}} = \pm\,\cdot 048 \tag{45}$$

Hence the absolute probable errors of $[\tau\cdot f]$, $[\mu\cdot e]$ are

$$[\tau\cdot f]\;\ldots\ldots\;\sqrt{(\cdot 048)^2 + (\cdot 098)^2 + (\cdot 037)^2 + (\cdot 029)^2}$$
$$[\mu\cdot e]\;\ldots\ldots\;\sqrt{(\cdot 056)^2 + (\cdot 061)^2 + (\cdot 048)^2 + (\cdot 021)^2}$$

And finally,

$$[\tau\cdot f] = \frac{474}{100}\,\mathrm{I} - 3\cdot 10 \pm \cdot 119 \tag{46}$$

$$[\mu\cdot e] = \frac{338}{100}\,\mathrm{I} - 1\cdot 24 \pm \cdot 098 \tag{47}$$

where $\mathrm{I} = \frac{1}{12}\,\mathbf{OF}$; and the small quantities are expressed in *millionths of a yard.*

10.

In the following Section, equation (7), it will be seen that the length $\mathbf{F}$ of the foot $[a\cdot p]$ expressed in terms of Standard Yard No. 55 is,

$$\mathbf{F} = \tfrac{1}{3}\,\mathbf{Y}_{55} - 0\cdot 36 + 0\cdot 0066\,(t - 62)$$

where $\mathbf{Y}_{55}$ is the length of the Standard Yard No. 55 at any temperature t, $\mathbf{OF}$ being at the same. Thence we find,

$$\frac{474}{1200}\,\mathbf{F} = \frac{474}{3600}\,\mathbf{Y}_{55} - 0\cdot 14 + \cdot 0026\,(t - 62)$$

with a probable error of $\pm$ ·043 (corresponding to $t = 62$, only) and

$$\frac{338}{1200}\,\mathbf{F} = \frac{338}{3600}\,\mathbf{Y}_{55} - 0{\cdot}10 + {\cdot}0019\,(t - 62)$$

with a probable error of $\pm$ ·030 (corresponding to $t = 62$, only).

Consequently,

$$[\tau \cdot f] = \frac{474}{3600}\,\mathbf{Y}_{55} - 3{\cdot}24 + {\cdot}0026\,(t - 62) \pm {\cdot}127$$

$$[\mu \cdot e] = \frac{338}{3600}\,\mathbf{Y}_{55} - 1{\cdot}34 + {\cdot}0019\,(t - 62) \pm {\cdot}102$$

where the probable errors have reference to the lengths at 62°.

The total number of micrometer readings made for the determination of the errors of subdivisions of **OF** is 8092.

V.

COMPARISON OF THE IRON FOOT

OF WITH $\mathbf{Y}_{55}$

In order to determine the true length of **OF** it was compared with Copy No. 55 of the Standard Yard.

The yard and foot were mounted close together in one box. The centres of the bars were opposite to one another, and their lengths parallel. Four microscopes were adjusted from the centre stone pier, having (1) their axes vertical, (2) their foci in one horizontal straight line, (3) the distances from centre to centre one foot exactly. The perfecting of all these adjustments takes a considerable amount of time and patience. First, the Standard Yard being made truly level on its rollers, the outer microscopes, H on the left, and K on the right, are levelled and adjusted to focus over the terminal lines of the yard. The yard is then removed, and a very fine silk thread is stretched tightly under the microscopes, being brought accurately to the focus of H and K, and made to pass through the intersection of the cross-hairs of those microscopes. The inner microscopes A and C are then by means of this silk thread brought into line, the intersection of their cross-hairs being made to bisect the thread: by this means very great accuracy may be attained. The distance of A from H, C from A, and K from C,—that is, the distance of the zeros or collimation centres,—is then made as nearly as possible one foot, by bringing **OF**, carefully levelled, first under H and A, then under A and C, then under C and K. Thus A and C are brought to their proper distance with respect to one another and to H and K. The levelling of A and C is then attended to. These different adjustments do of necessity disturb one another, and have to be repeated each several times; but by patience they may be made very perfect. The microscopes finally are left in such position that only small positive readings are required to be made in the observations of comparison.

The microscopes H and A (on the left) have their micrometer heads to the left; C and K (on the right) have their micrometer heads to the right. The line of the microscopes is as nearly as practicable parallel to the line of rails. The values of one division of the microscopes are—

H	A	C	K
0·7949	1·177	0·869	0·7980

The Standard Yard has two thermometers in its wells, and the Foot (**OF**) one only.

The bars are visited three times generally during the day; the method of observing being as follows: (*a*) The thermometers are first read; (*b*) the yard is adjusted under H and K, and these microscopes read three times; (*c*) **OF** is adjusted under H and A, and these microscopes read, three readings; (*d*) **OF** is adjusted under A and C, and these microscopes read, three readings; (*e*) **OF** is adjusted under C and K, and these microscopes read, three readings; (*f*) **SY** is again adjusted under H and K, and these microscopes read, three readings; (*g*) the three thermometers are again read, which closes the operation.

The "adjustments" referred to in this place are effected by the transverse and longitudinal motions of the two carriages, whereby the terminal lines of the bar are brought into precisely their proper places in the fields of view of the microscopes. Vertical adjustment of the bars, if necessary (for focus), is effected by the levelling keys. The box in which the bars lie remains closed during the whole operation, the lines and thermometers being observed through the apertures in the cover.

The whole of the readings specified above, taken in one visit, are made as rapidly as is consistent with perfect accuracy.

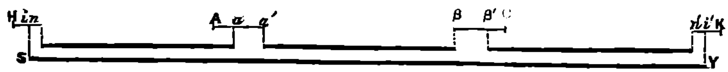
As a specimen page of the observation book, showing the readings taken in *one visit*, the following is subjoined :—

Date.	Bar.	Microscope Readings.		Thermometers.	
				OF	$\mathbf{Y}_{55}$
Nov. 6	$\mathbf{Y}_{55}$	H	K		
9 A.M.		12·1	5·4	55·70	55·62
		12·1	5·7		55·62
		11·8	5·1		
	OF	H	A		
		12·2	5·0		
		12·4	5·5		
		11·8	5·0		
	OF	A	C		
		4·7	8·8		
		5·3	8·7		
		5·1	8·9		
	OF	C	K		
		9·1	7·0		
		9·5	6·1		
		9·4	5·9		
	$\mathbf{Y}_{55}$	H	K		
		10·6	5·6		
		10·9	5·9		55·67
		10·4	5·0	55·75	55·67

Let now $r_1\, r'_1$ represent the mean of the first (three) readings of H and K on $\mathbf{Y}_{55}$, $r_2\, r'_2$ the mean of the second readings at the close: $r\, \varrho'_1$ the mean reading of H and A, $\varrho_1\, \varrho'_2$ of A and C, $\varrho_2\, r'$ of C and K; so that the above Table stands thus :—

$$r_1\ r'_1$$
$$r\ \varrho'_1$$
$$\varrho_1\ \varrho'_2$$
$$\varrho_2\ r'$$
$$r_2\ r'_2$$

Now in the diagram let **SY** be the yard and the three lines parallel to it the three positions taken up by **OF**. Let H, A, C, K, represent the positions of the zeros of the microscopes.



Then if Z be the distance from H to K, $\mathbf{Y}_{55}$ the length of the yard, **F** the length of the foot,

$$Z = Hi + \mathbf{Y}_{55} + i'K$$
$$Z = Hn + \alpha\alpha' + 3\,\mathbf{F} + \beta\beta' + n'K$$

Now if h_1, a_1, c_1, k_1, be the absolute lengths of one division in the micrometers H, A, C, K,

$$Hi + i'K = \tfrac{1}{2}(r_1 + r_2)\,h + \tfrac{1}{2}(r'_1 + r'_2)\,k$$
$$Hn = rh$$
$$\alpha\alpha' = (\varrho_1 - \varrho'_1)\,a$$
$$\beta\beta' = (\varrho'_2 - \varrho_2)\,c$$
$$n'K = r'k$$

Consequently we have the equations—

$$Z = Y_{55} + \tfrac{1}{2}(r_1 + r_2)\,h + \tfrac{1}{2}(r'_1 + r'_2)\,k$$
$$Z = 3\,F + rh + (\varrho_1 - \varrho'_1)\,a + (\varrho'_2 - \varrho_2)\,c + r'k$$
$$-Y_{55} + 3\,F = \left(\frac{r_1 + r_2}{2} - r\right)h + (\varrho'_1 - \varrho_1)\,a + (\varrho_2 - \varrho'_2)\,c + \left(\frac{r'_1 + r_2}{2} - r'\right)k$$
$$\text{or } F - \tfrac{1}{3}Y_{55} = \tfrac{1}{3}\left(\frac{r_1 + r_2}{2} - r\right)h + \left(\frac{\varrho'_1 - \varrho_1}{3}\right)a + \left(\frac{\varrho_2 - \varrho'_2}{3}\right)c + \tfrac{1}{3}\left(\frac{r'_1 + r'_2}{2} - r'\right)k$$

In this shape the observations are brought together in the following Table, each line representing the result of one visit or one comparison. The number of comparisons is thirty, extending over twelve days. The total number of micrometer readings 900.

COMPARISON of **OF** with the STANDARD YARD No. 55.

Date.	Temperature.	Mean Micrometer Readings.	$F - \frac{1}{3}Y_{55}$
1863.			
November 4	56°·38	− 1·28 *h* + 0·50 *a* + 0·17 *c* − 1·13 *k*	− 1·18
„ „	56·41	− 0·05 *h* − 0·17 *a* + 0·38 *c* − 0·35 *k*	− 0·19
„ 5	55·51	+ 0·18 *h* − 0·03 *a* − 0·02 *c* − 1·05 *k*	− 0·75
„ „	55·66	− 0·02 *h* − 0·31 *a* − 0·23 *c* − 0·50 *k*	− 0·97
„ „	56·00	+ 0·18 *h* − 0·18 *a* + 0·26 *c* + 0·27 *k*	+ 0·38
„ 6	55·69	− 0·27 *h* + 0·05 *a* + 0·18 *c* − 0·29 *k*	− 0·23
„ „	55·84	+ 0·62 *h* + 0·34 *a* − 0·08 *c* − 0·92 *k*	+ 0·10
„ „	55·96	− 0·27 *h* − 0·54 *a* − 0·01 *c* + 0·76 *k*	− 0·25
„ 7	55·47	+ 0·32 *h* − 0·16 *a* + 0·19 *c* − 1·06 *k*	− 0·61
„ „	55·56	− 0·07 *h* + 0·12 *a* − 0·28 *c* − 0·56 *k*	− 0·60
„ 8	55·66	+ 0·25 *h* − 0·11 *a* + 0·50 *c* − 0·79 *k*	− 0·13
„ 9	55·91	+ 0·00 *h* − 0·12 *a* − 0·41 *c* + 0·02 *k*	− 0·48
„ „	55·04	+ 0·78 *h* + 0·28 *a* − 0·77 *c* − 1·37 *k*	− 0·81
„ „	55·34	− 0·25 *h* − 0·06 *a* − 0·01 *c* + 0·54 *k*	+ 0·15
„ 10	54·76	− 0·02 *h* − 0·02 *a* + 0·09 *c* − 0·76 *k*	− 0·56
„ „	54·46	+ 0·12 *h* + 0·25 *a* − 0·52 *c* − 0·51 *k*	− 0·47
„ „	54·44	− 0·01 *h* + 0·07 *a* − 0·96 *c* + 0·42 *k*	− 0·42
„ 11	53·44	+ 0·55 *h* − 0·49 *a* − 0·61 *c* + 0·59 *k*	− 0·20
„ „	53·47	− 0·74 *h* + 0·03 *a* − 0·21 *c* − 0·31 *k*	− 0·98
„ „	53·43	+ 0·32 *h* − 0·41 *a* − 0·33 *c* + 0·61 *k*	− 0·03
1864.			
February 8	38·84	+ 0·27 *h* − 3·30 *a* + 1·59 *c* + 2·65 *k*	− 0·18
„ 9	36·72	− 0·27 *h* − 2·15 *a* − 2·43 *c* + 5·65 *k*	− 0·36
„ „	36·72	− 1·40 *h* − 0·34 *a* + 0·50 *c* − 0·67 *k*	− 1·61
„ „	36·97	− 0·58 *h* − 0·08 *a* + 0·29 *c* + 0·71 *k*	+ 0·26
„ „	37·18	+ 2·54 *h* − 3·24 *a* − 0·82 *c* + 2·55 *k*	− 0·47
„ „	37·10	+ 1·45 *h* − 0·13 *a* − 0·06 *c* − 1·09 *k*	+ 0·08
„ 10	36·13	− 0·07 *h* − 0·83 *a* + 0·55 *c* + 0·83 *k*	+ 0·11
„ „	36·51	+ 0·12 *h* − 0·51 *a* + 0·13 *c* − 0·62 *k*	− 0·89
„ „	36·72	− 0·15 *h* − 0·06 *a* − 0·01 *c* − 1·35 *k*	− 1·27
„ 11	37·07	− 0·16 *h* − 0·67 *a* − 1·31 *c* + 1·44 *k*	− 0·91

Each of the quantities in the column headed "Temperature" is the mean of six thermometer readings, corrected for the errors of the three thermometers employed. The fourth column gives the difference of length $\mathsf{F} - \frac{1}{3}\,\mathbf{Y}_{55}$ expressed in *millionths of a yard.*

In order to obtain the difference of length at 62° and at any other temperature, let $\mathsf{F} - \frac{1}{3}\,\mathbf{Y}_{55}$ be expressed by

$$x + (t - 62)\,y \qquad (1)$$

where x is the difference $\mathsf{F} - \frac{1}{3}\,\mathbf{Y}_{55}$ at 62°, and y the relative expansion of F over $\frac{1}{3}\,\mathbf{Y}_{55}$ for each degree Fahrenheit. If we compare this expression with the thirty differences of length at thirty different temperatures given in the Table, there result thirty equations for the determination of x and y. These equations, treated by the method of least squares, give

$$\begin{aligned} 30\,x - 385{\cdot}16\,y + 13{\cdot}47 &= 0 \\ -\,385{\cdot}61\,x + 7192{\cdot}15\,y - 187{\cdot}88 &= 0 \end{aligned} \qquad (2)$$

If we write A and B for the absolute terms of these equations, and eliminate x and y, we get

$$\begin{aligned} x + 0{\cdot}107234\,\mathrm{A} + 0{\cdot}005749\,\mathrm{B} &= 0 \\ y + 0{\cdot}005749\,\mathrm{A} + 0{\cdot}000447\,\mathrm{B} &= 0 \end{aligned} \qquad (3)$$

Substituting the numerical values of A and B,

$$\begin{aligned} x &= -\,0{\cdot}364 \\ y &= +\,0{\cdot}0066 \end{aligned} \qquad (4)$$

Now for the probable errors we recur to the following theorem:—If there be a series of equations

$$\begin{aligned} a_1\,x + b_1\,y + n_1 &= 0 \\ a_2\,x + b_2\,y + n_2 &= 0 \\ a_3\,x + b_3\,y + n_3 &= 0 \\ \vdots\qquad\qquad & \quad\vdots \\ a_i\,x + b_i\,y + n_i &= 0 \end{aligned}$$

from which x and y are to be obtained, then, by the method of least squares,

$$0 = x + \frac{(b^2)}{(a^2)\,(b^2) - (ab)^2}\cdot(an) - \frac{(ab)}{(a^2)\,(b^2) - (ab)^2}\cdot(bn)$$

$$0 = y - \frac{(ab)}{(a^2)\,(b^2) - (ab)^2}\cdot(an) + \frac{(a^2)}{(a^2)\,(b^2) - (ab)^2}\cdot(bn)$$

Then the sum of the squares of the coefficients of the symbols $n_1\,n_2\,\ldots\,n_i$ in the development of $x + fy$ is,

$$\frac{(b^2) - 2\,(ab)\,f + (a^2)\,f^2}{(a^2)\,(b^2) - (ab)^2} \qquad (5)$$

which is also the reciprocal of the weight of the determination of $x + fy$.

Returning to the case under consideration, we see from (3),

Reciprocal of weight of	x	......	0·10723
,, ,,	y	......	0·000447
,, ,,	$x + fy$	......	$0{\cdot}10723 + {\cdot}01150f + {\cdot}000447f^2$.

By substituting the values of x and y in the 30 equations from which they are obtained, the errors are found, and the sum of their squares = 6·734; hence the probable errors of x and y are,

$$\text{for } x \ldots\ldots \pm \cdot 674 \sqrt{\frac{6 \cdot 734}{30 - 2}} \sqrt{\cdot 1072} = \pm 0 \cdot 108 \qquad (6)$$

$$\text{for } y \ldots\ldots \pm \cdot 674 \sqrt{\frac{6 \cdot 734}{30 - 2}} \sqrt{\cdot 00045} = \pm 0 \cdot 0070$$

Finally, for the length of the foot **OF** we have,

$$\mathbf{F} = \tfrac{1}{3}\, \mathbf{Y}_{65} - 0 \cdot 36 + 0 \cdot 0066\, (t - 62) \qquad (7)$$

The probable errors of this determination for the temperature 62° + f° Fahrenheit being,

$$(\cdot 011715 + \cdot 001256 f + \cdot 0000488 f^2)^{\frac{1}{2}} \qquad (8)$$

The mean increase of temperature during a visit to the bars in this series of comparisons—that is, the mean value of the excess of the *second* readings of the thermometers above the *first* readings—is 0°·06, being the effect of the presence of the observer.

VI.

DETERMINATION OF THE ABSOLUTE EXPANSIONS

OF TWO BARS

OI_1 AND OI_2

We shall here record some carefully conducted experiments whereby the absolute expansions of two ten-feet iron bars OI_1 and OI_2 were obtained.

The observations were made in November 1857, when the temperature of the external air varied from 40° to 50°. The bar was observed about 9 A.M., at the temperature of the Bar Room, having lain under the microscopes during the preceding night. The micrometer readings being taken and registered, the bar was removed to another room in which the temperature was about 100°. Here the bar was allowed to remain until it acquired a steady temperature, as ascertained by the readings of the thermometers in the bar. It was then carefully enveloped in blankets, and with all practicable expedition taken to the Bar Room and adjusted under the microscopes. Observations were then taken at intervals of time, the microscopes being read simultaneously by two observers, and the thermometers by a third.

The following Table contains the observations on OI_1. The value of one division of the West micrometer = 0·7738, and that of the East micrometer = 1·1473 (*millionths of a yard*):—

Date.	Temperature, Fahrenheit.	Microscope Readings.		$OI_1 - z$	Date.	Temperature, Fahrenheit.	Microscope Readings.		$OI_1 - z$
		West.	East.				West.	East.	
1857.	°	d	d		1857.	°	d	d	
Nov. 19	52·9	150	59	44·4	Nov. 25	45·6	96	120	−63·4
	97·0	1427	114	973·4		74·0	914	149	536·3
	94·0	1399	147	913·9		71·0	862	166	476·6
	91·0	1362	168	861·2		68·0	817	186	418·8
„ 20	49·5	70	92	−51·4	„ 26	44·0	402	344	−83·6
	100·0	1501	108	1037·6		118·0	1925	28	1457·4
	97·0	1446	123	977·8		115·0	1864	47	1388·4
	94·0	1399	135	927·7		112·0	1804	71	1314·5

The 5th and 10th columns show the excess of the length of the bar above the distance of the zeros of the microscopes, obtained from the microscope readings in the preceding columns.

If we reduce each of these groups by the method of least squares we obtain the following results for the different days :—

Nov. 19th ;	Expansion for 1°	=	21·096
„ 20th ;	„ „	=	21·725
„ 25th ;	„ „	=	21·269
„ 26th ;	„ „	=	20·713

Reducing the whole by the method of least squares, and then summing the squares of the errors of the individual measures, we have,

Expansion of $\mathbf{OI}_1$ *for* 1° *Fahrenheit* = 21·055 ± ·089

The observations on $\mathbf{OI}_2$ are given in the next Table :—

Date.	Temperature, Fahrenheit.	Microscope Readings.		$\mathbf{OI}_2 - z$	Date.	Temperature, Fahrenheit.	Microscope Readings.		$\mathbf{OI}_2 - z$
		West.	East.				West.	East.	
1857.	°	d	d		1857.	°	d	d	
Nov. 2	51·5	232	251	−109·0	Nov. 13	43·1	194	348	−249·1
	82·0	1150	284	563·6		95·2	1620	350	852·0
	79·2	1011	257	487·4		91·8	1489	317	788·5
	73·5	836	260	348·7		88·5	1471	356	729·8
„ 3	54·9	552	417	− 51·3	„ 14	45·0	192	312	−209·4
	80·2	1234	395	501·7		97·0	1572	269	907·8
	77·6	1152	381	454·3		94·0	1538	292	855·1
	73·8	926	295	378·4		91·0	1492	314	794·3
„ 4	55·6	631	463	− 43·5	„ 16	45·3	212	304	−184·7
	84·3	1382	447	556·6		93·0	1522	320	810·6
	80·8	1239	408	490·6		90·0	1492	349	754·1
	76·8	1142	413	409·8		87·0	1461	379	695·7
„ 12	47·5	207	280	−161·1	„ 18	51·4	244	346	−208·2
	85·9	1274	289	654·2		102·0	1604	331	861·4
	81·5	1176	294	572·7		99·0	1564	350	808·7
	77·0	1083	305	488·1		96·0	1523	369	755·1

The 5th and 10th columns show the excess of the length of the bar above the distance of the zeros of the microscopes, obtained from the microscope readings in the two preceding columns.

If from each group we obtain by the method of least squares the value of the expansion of the bar for 1°, we get the following values :—

November 2	..... 21·595	21·316	November 13.
„ 3	..... 22·206	21·662	„ 14.
„ 4	..... 21·116	20·983	„ 16.
„ 12	..... 21·539	21·347	„ 18.

If we reduce the whole of these observations in one mass, we have the following final result :—

Expansion of $\mathbf{OI}_2$ *for* 1° *Fahrenheit* = 21·400 ± ·050

These determinations may be considered tolerably satisfactory. In two respects they are, however, not unexceptionable: 1st, the bar when observed at the high temperatures was not in a state of quiescence and equilibrium, but in the state of cooling. Everything was done which the means then at disposal permitted, to retard the cooling, as wrapping in blankets, &c., nevertheless the temperature of the bar was in a changing state, and it is assumed that the thermometer, as read at any instant, indicated the temperature of the metal at that instant; 2d, the interval of time between the first observation in the morning at low natural temperature and the observations at high temperature was from two to three hours. Was there any alteration in the distance of the microscopes in that time? This point can be satisfactorily answered, as the Ordnance Standard O_1 was in the Bar Room all the time, and was used to test the distance of the microscopes. It may be asserted that there were no changes in the distance of the zeros of the microscopes of sufficient magnitude to influence the results.

The expansions obtained above may be otherwise thus stated:—

Expansion for 10° Fahrenheit.	
OI_1	OI_2
210·55 ± ·89	214·00 ± ·50

These expansions differ but little from the expansions of the Ordnance Standard O_1 as shown in the "*Account of the Measurement of the Lough Foyle Base.*" The absolute expansion of this bar, as determined by the experiments of the late Captain Drummond, R.E., in 1827, was 21·740 per 1° Fahrenheit: but the values resulting from four different series of comparisons in 1844, 1845, 1846, gave 20·33, 20·23, 20·65, 19·74, of which the mean is 20·26. The expansion of O_1 as inferred from the comparisons made between it and the compensation bars during the measurement of the Salisbury Plain Base gave 21·23. This differs but one-hundredth part from the expansion of either OI_1 or OI_2 as determined in these experiments.

From the probable errors exhibited in the above Table it would appear that if the length of OI_1 or OI_2 were observed at 52° or 72°, their lengths at 62° might be inferred with a probable error of less than one millionth of a yard.

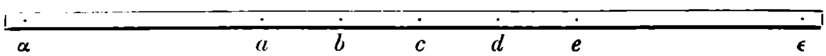
VII.

DETERMINATION OF THE LENGTH OF THE ORDNANCE STANDARD

$\mathbf{O}_1$

1.

The determination of the length of this Standard in terms of the yard $\mathbf{Y}_{55}$ is necessary in order to express the length of any of the sides in the Triangulation of Great Britain and Ireland, as published in the "*Account of the Principal Triangulation*," in terms of the National Standard of length. The determination involves the use of an intermediate bar, carrying points on its surface, for the standard $\mathbf{O}_1$ has no subdivisions, the extreme dots being at the mid-depth or neutral axis of the bar. The bar used intermediately is designated $\mathbf{OI}_1$ (Ordnance Intermediate, No. 1.), and has been already described as having seven points on its upper surface distributed thus:



and which we shall for convenience designate by the letters *α a b c d e ε*. The extreme spaces $[\alpha \cdot a]$, $[e \cdot \epsilon]$ are each one yard; the four central spaces are each one foot.

The comparisons made were the following:—

(1)	$[\alpha \cdot a]$	*with Standard Yard* $\mathbf{Y}_{55}$
(2)	$[a \cdot d]$	,, ,,
(3)	$[b \cdot e]$	,, ,,
(4)	$[e \cdot \epsilon]$	,, ,,
(5)	$[a \cdot b]$	*with Ordnance Foot* **OF**
(6)	$[d \cdot e]$	,, ,,
(7)	$[\alpha \cdot \epsilon]$	*with Ordnance Standard* $\mathbf{O}_1$

(1) The left yard $[\alpha \cdot a]$ was compared with the standard yard $\mathbf{Y}_{55}$ on May 19th, 20th, 21st, June 8th, 25th, 27th; in all twenty comparisons; each "comparison" being the result of one visit.

(2) The left centre yard $[a \cdot d]$ was compared with $\mathbf{Y}_{55}$ on February 22d, 23d, May 23d, 25th, June 9th, 29th; twenty-one comparisons.

(3) The right centre yard $[b \cdot e]$ was compared with $\mathbf{Y}_{55}$ on February 23d, 24th, 25th, May 26th, 27th, 28th, June 10th, 17th, 29th, 30th; twenty-two comparisons.

(4) The right yard $[e \cdot \epsilon]$ was compared with $\mathbf{Y}_{55}$ on May 30th, 31st, June 1st, 18th, 30th, July 1st, 2d; in all twenty comparisons.

(5) The left centre foot $[a \cdot b]$ was compared with **OF** on June 2d and 20th, July 4th; ten comparisons.

(6) The right centre foot [$d \cdot e$] was compared with **OF** on June 21st, 22d, July 4th; ten comparisons.

With the exception of the comparisons of the two central yards in February, the remainder of the comparisons in May, June, and July were made at temperatures differing but little from the standard temperature of 62°. The comparisons in February were made at the temperature of about 38°, and serve to establish the difference of expansions of $\mathbf{OI}_1$ and $\mathbf{Y}_{55}$.

(7) The ten-feet space [$\alpha \cdot \epsilon$] was compared with $\mathbf{O}_1$ on February 18th, 19th, 20th, May 16th, 17th, 18th, June 4th, 6th, 7th, 23d, 24th; in all forty comparisons. The comparisons in May and June were made at temperatures close to 62°, those in February at the temperature of 40°. By this arrangement the difference of the rates of expansion of $\mathbf{OI}_1$ and $\mathbf{O}_1$ become well ascertained.

The total number of micrometer readings in these seven series of comparisons is 2708.

From September 1863 until March 1864 the number of micrometer readings taken at a single *visit* and considered as *one comparison* of two bars A, B, was 36, disposed as follows:—

Bar.	Microscopes.	
	H	**K**
A	3 readings.	3 readings.
B	3 "	3 "
B	3 "	3 "
A	3 "	3 "
A	3 "	3 "
B	3 "	3 "

The experience gained in that time served to show that the number of readings so taken was unnecessarily large, and that a smaller number of readings would be almost equally valuable, while there would be the advantage of less time occupied in a comparison, and less heat imparted to the bar by the lighted candles and the person of the observer. Consequently in comparisons made on this bar after the commencement of May 1864 the number of readings of micrometers in a single visit is as follows:—

Bar.	Microscopes.	
	H	**K**
A	2 readings.	2 readings.
B	2 "	2 "
B	2 "	2 "
A	2 "	2 "

Here one comparison involves 16 micrometer readings. These, together with the eight thermometer readings, occupy about ten minutes.

No alteration has been made in the number of visits during the course of one day. In general four visits or comparisons are made at hours as far apart as practicable.

During the comparisons now under consideration, as in all others, the two bars being compared are placed side by side in the same box. The bar $\mathbf{O}_1$ rests on rollers (as it has always done) at one fourth and three fourths of its length. Two thermometers are placed in each bar (excepting in the case of **OF**, which takes only one), and are read at the commencement and close of each visit. The box is kept closed during the whole operation, and it has been already explained that the thermometers are read through apertures in the lid of the box, having sliding covers; and the focal adjustment of the bars can also be effected without removing the cover of the box. This last remark, however, does not apply to the bar $\mathbf{O}_1$. Its rollers, fixed in the bottom of the box and adjusted so that the bar shall be truly horizontal, are not provided with any elevating screw for adjusting to focus; but at the commencement of a series of comparisons of $\mathbf{O}_1$ after it has been ascertained that this bar is truly level, the microscopes are adjusted to focus over its dots.

It has been often remarked by the observers that a bar once in focus under the microscopes will remain perfectly in focus for any length of time,—a remarkable proof of the perfect steadiness and rigidity of the whole apparatus. Also, if a box containing two bars whose upper surfaces are perfectly level be removed from the carriages and be replaced some months after, (the carriages standing on the same part of the iron rails as before), it has been invariably found that these bars are still perfectly level.

The extremities of the measure of ten feet on $\mathbf{O}_1$ are indicated, not by lines but by dots. Considering the number of years this standard has been in use the dots retain their circular shape well, yet not perfectly. They are also very large, and the diameter of either measures one hundred divisions of the micrometers H or K; so that to observe the centre with the nicety one would desire is difficult. A large proportion of the probable error of our results will doubtless arise from this cause.

Nor are the lines on $\mathbf{OI}_1$ very good; they are about 30 divisions of the micrometer in breadth; but the edges are straight and parallel, and this is of greater importance than the breadth.

The following Tables contain the observations in detail, each line corresponding to one visit.

LEFT YARD.

Date. 1864.	Temp.	$\mathbf{OI}_1$	$\mathbf{Y}_{55}$	Difference of Length in Micrometer Divisions.	$[a \cdot a] - \mathbf{Y}_{55}$
May 19	64°·31	93·35 *h* + 70·95 *k*	113·38 *h* + 119·55 *k*	20·03 *h* + 48·60 *k*	+54·70
" "	64·31	85·93 *h* + 78·25 *k*	114·13 *h* + 116·98 *k*	28·20 *h* + 38·73 *k*	53·32
" 20	64·13	83·38 *h* + 83·55 *k*	126·15 *h* + 107·70 *k*	42·77 *h* + 24·15 *k*	53·27
" "	64·23	84·63 *h* + 83·20 *k*	116·10 *h* + 120·83 *k*	31·47 *h* + 37·63 *k*	55·04
" "	64·31	87·53 *h* + 79·08 *k*	111·10 *h* + 124·25 *k*	23·57 *h* + 45·17 *k*	54·78
" "	64·40	78·03 *h* + 87·85 *k*	119·28 *h* + 114·78 *k*	41·25 *h* + 26·93 *k*	54·28
" 21	64·50	91·25 *h* + 76·98 *k*	116·98 *h* + 118·85 *k*	25·73 *h* + 41·87 *k*	53·87
" "	64·53	85·23 *h* + 81·85 *k*	122·20 *h* + 114·50 *k*	36·97 *h* + 32·65 *k*	55·44
June 8	62·92	38·68 *h* + 35·65 *k*	76·10 *h* + 67·10 *k*	37·42 *h* + 31·45 *k*	54·84
" "	63·07	30·48 *h* + 42·28 *k*	69·98 *h* + 72·88 *k*	39·50 *h* + 30·60 *k*	55·82
" "	63·19	33·25 *h* + 40·53 *k*	73·25 *h* + 68·85 *k*	40·00 *h* + 28·32 *k*	54·40
" "	63·32	48·05 *h* + 23·70 *k*	83·13 *h* + 59·13 *k*	35·08 *h* + 35·43 *k*	56·16
" 25	61·88	24·00 *h* + 27·85 *k*	57·73 *h* + 62·03 *k*	33·73 *h* + 34·18 *k*	54·09
" "	61·97	30·10 *h* + 19·85 *k*	63·45 *h* + 55·83 *k*	33·35 *h* + 35·98 *k*	55·22
" "	62·02	23·35 *h* + 26·18 *k*	65·88 *h* + 53·40 *k*	42·53 *h* + 27·22 *k*	55·53
" "	62·09	26·98 *h* + 21·50 *k*	68·10 *h* + 49·95 *k*	41·12 *h* + 28·45 *k*	55·39
" 27	61·64	25·75 *h* + 28·30 *k*	69·07 *h* + 53·60 *k*	43·32 *h* + 25·30 *k*	54·62
" "	61·67	23·40 *h* + 31·73 *k*	55·75 *h* + 65·83 *k*	32·35 *h* + 34·10 *k*	52·93
" "	61·75	31·80 *h* + 19·98 *k*	63·88 *h* + 56·95 *k*	32·08 *h* + 36·97 *k*	55·00
" "	62·02	21·33 *h* + 28·55 *k*	59·28 *h* + 60·15 *k*	37·95 *h* + 31·60 *k*	+55·38

LEFT CENTRE YARD.

Date. 1864.	Temp.	Ol_1	Y_{55}	Difference of Length in Micrometer Divisions.	$[a \cdot d] - Y_{55}$
Feb. 22	38°·61	47·79 h + 40·46 k	22·47 h + 10·20 k	25·32 h + 30·26 k	−44·27
„ „	38·79	43·11 h + 45·59 k	12·54 h + 20·66 k	30·57 h + 24·93 k	44·19
„ „	38·83	45·77 h + 41·66 k	14·79 h + 16·91 k	30·98 h + 24·75 k	44·38
„ 23	38·51	50·85 h + 39·89 k	19·10 h + 14·64 k	31·75 h + 25·25 k	45·39
„ „	38·48	47·68 h + 42·01 k	11·81 h + 22·20 k	35·87 h + 19·81 k	44·32
May 23	63·58	107·85 h + 90·73 k	70·50 h + 69·25 k	37·35 h + 21·48 k	46·83
„ „	63·67	107·95 h + 89·23 k	72·38 h + 66·43 k	35·57 h + 22·80 k	46·47
„ „	63·74	102·93 h + 100·38 k	70·15 h + 74·23 k	32·78 h + 26·15 k	46·92
„ „	63·74	99·25 h + 105·20 k	70·23 h + 75·00 k	29·02 h + 30·20 k	47·17
„ 25	62·04	110·03 h + 102·95 k	74·10 h + 78·65 k	35·93 h + 24·30 k	47·95
„ „	62·10	95·98 h + 117·60 k	67·38 h + 89·03 k	28·60 h + 28·57 k	45·53
„ „	62·11	100·95 h + 111·80 k	68·80 h + 84·40 k	32·15 h + 27·40 k	47·42
„ „	62·15	102·60 h + 109·70 k	72·68 h + 80·58 k	29·92 h + 29·12 k	47·02
June 9	63·87	52·83 h + 54·73 k	26·28 h + 20·55 k	26·55 h + 34·18 k	48·38
„ „	63·97	60·30 h + 47·30 k	32·20 h + 17·98 k	28·10 h + 29·32 k	45·73
„ „	64·05	65·53 h + 42·23 k	32·60 h + 15·83 k	32·93 h + 26·40 k	47·24
„ „	64·16	46·08 h + 60·08 k	23·30 h + 24·13 k	22·78 h + 35·95 k	46·80
„ 29	61·26	69·75 h + 57·90 k	35·28 h + 33·88 k	34·47 h + 24·02 k	46·57
„ „	61·28	61·80 h + 62·88 k	33·20 h + 35·25 k	28·60 h + 27·63 k	44·78
„ „	61·30	63·05 h + 63·53 k	32·28 h + 34·93 k	30·77 h + 28·60 k	47·28
„ „	61·55	70·93 h + 56·28 k	34·10 h + 32·87 k	36·83 h + 23·41 k	−47·96

RIGHT CENTRE YARD.

Date. 1864.	Temp.	Ol_1	Y_{55}	Difference of Length in Micrometer Divisions.	$[b \cdot e] - Y_{55}$
Feb. 23	38°·67	23·20 h + 23·19 k	14·79 h + 17·73 k	8·41 h + 4·46 k	−10·24
„ 24	38·28	19·97 h + 27·48 k	14·53 h + 19·24 k	5·44 h + 8·24 k	10·90
„ „	38·20	26·31 h + 22·36 k	23·91 h + 11·74 k	2·40 h + 10·62 k	10·38
„ „	38·25	18·61 h + 29·79 k	20·68 h + 14·91 k	−2·07 h + 14·88 k	10·22
„ „	38·27	28·20 h + 19·20 k	18·83 h + 15·33 k	9·37 h + 3·87 k	10·54
„ 25	38·04	21·49 h + 27·42 k	16·77 h + 19·65 k	4·72 h + 7·77 k	9·95
May 26	61·84	80·38 h + 88·20 k	79·38 h + 79·60 k	1·00 h + 8·60 k	7·66
„ „	61·84	88·70 h + 80·40 k	80·40 h + 74·88 k	8·30 h + 5·52 k	11·00
„ 27	60·89	95·03 h + 76·95 k	89·00 h + 71·30 k	6·03 h + 5·65 k	9·30
„ „	60·86	83·68 h + 90·58 k	73·95 h + 89·33 k	9·73 h + 1·25 k	8·73
„ „	60·85	87·08 h + 86·45 k	76·23 h + 86·00 k	10·85 h + 0·45 k	8·98
„ 28	60·28	84·85 h + 91·38 k	78·73 h + 84·78 k	6·12 h + 6·60 k	10·13
„ „	60·31	74·80 h + 102·58 k	73·55 h + 92·63 k	1·25 h + 9·95 k	8·93
„ „	60·31	89·05 h + 87·55 k	89·63 h + 74·58 k	−0·58 h + 12·97 k	9·89
June 10	64·22	33·33 h + 30·85 k	23·80 h + 25·93 k	9·53 h + 4·92 k	11·50
„ 17	61·55	52·73 h + 52·85 k	45·38 h + 46·00 k	7·35 h + 6·85 k	11·31
„ „	61·50	54·90 h + 52·08 k	52·53 h + 41·48 k	2·37 h + 10·60 k	10·34
„ „	61·49	69·98 h + 37·55 k	53·53 h + 39·70 k	16·45 h − 2·15 k	11·36
„ 29	61·79	81·05 h + 88·35 k	83·48 h + 70·50 k	−2·43 h + 17·85 k	12·31
„ 30	61·32	79·60 h + 90·85 k	77·43 h + 81·78 k	2·17 h + 9·07 k	8·96
„ „	61·36	81·53 h + 88·45 k	75·13 h + 82·05 k	6·40 h + 6·40 k	10·19
„ „	61·44	78·83 h + 91·78 k	76·05 h + 82·18 k	2·78 h + 9·60 k	− 9·87

RIGHT YARD.

Date. 1864.	Temp.	OI_1	Y_{85}	Difference of Length in Micrometer Divisions.	$[e \cdot s] - Y_{85}$
May 30	59·14°	26·95 *h* + 41·65 *k*	23·95 *h* + 32·28 *k*	3·00 *h* + 9·37 *k*	− 9·86
„ „	59·14	37·18 *h* + 32·58 *k*	25·13 *h* + 32·43 *k*	12·05 *h* + 0·15 *k*	9·70
„ „	59·14	31·08 *h* + 38·65 *k*	27·98 *h* + 29·40 *k*	3·10 *h* + 9·25 *k*	9·85
„ 31	58·63	27·20 *h* + 45·05 *k*	25·98 *h* + 31·95 *k*	1·22 *h* + 13·10 *k*	11·42
„ „	58·65	39·58 *h* + 31·18 *k*	32·08 *h* + 28·00 *k*	7·50 *h* + 3·18 *k*	8·50
„ „	58·65	36·10 *h* + 35·98 *k*	25·70 *h* + 34·78 *k*	10·40 *h* + 1·20 *k*	9·22
„ „	58·68	32·13 *h* + 38·75 *k*	29·03 *h* + 29·88 *k*	3·10 *h* + 8·87 *k*	9·54
June 1	58·04	30·50 *h* + 44·08 *k*	28·13 *h* + 35·03 *k*	2·37 *h* + 9·05 *k*	9·11
„ 18	61·40	30·40 *h* + 33·73 *k*	21·35 *h* + 30·08 *k*	9·05 *h* + 3·65 *k*	10·11
„ „	61·48	23·98 *h* + 37·53 *k*	18·00 *h* + 33·00 *k*	5·98 *h* + 4·53 *k*	8·37
„ „	61·54	27·73 *h* + 34·95 *k*	29·28 *h* + 21·55 *k*	−1·55 *h* + 13·40 *k*	9·46
„ „	61·62	25·88 *h* + 36·55 *k*	24·90 *h* + 26·05 *k*	0·98 *h* + 10·50 *k*	9·16
„ 30	61·65	67·35 *h* + 102·38 *k*	67·38 *h* + 87·90 *k*	−0·03 *h* + 14·48 *k*	11·53
July 1	61·51	70·88 *h* + 98·20 *k*	78·03 *h* + 78·15 *k*	−7·15 *h* + 20·05 *k*	10·32
„ „	61·53	85·13 *h* + 85·18 *k*	78·43 *h* + 77·53 *k*	6·70 *h* + 7·65 *k*	11·43
„ „	61·59	85·30 *h* + 83·50 *k*	75·40 *h* + 79·60 *k*	9·90 *h* + 3·90 *k*	10·98
„ 2	61·35	86·35 *h* + 84·43 *k*	79·78 *h* + 77·13 *k*	6·57 *h* + 7·30 *k*	11·05
„ „	61·39	80·20 *h* + 89·83 *k*	76·85 *h* + 80·15 *k*	3·35 *h* + 9·68 *k*	10·39
„ „	61·51	77·45 *h* + 93·28 *k*	71·43 *h* + 84·73 *k*	6·02 *h* + 8·55 *k*	11·61
„ „	61·58	80·85 *h* + 88·53 *k*	74·10 *h* + 80·55 *k*	6·75 *h* + 7·98 *k*	−11·73

LEFT FOOT.

Date. 1864.	Temp.	OI_1	OF	Difference of Length in Micrometer Divisions.	$[a \cdot b] - F$
June 2	57·67°	31·30 *h* + 26·85 *k*	20·60 *h* + 22·25 *k*	10·70 *h* + 4·60 *k*	−12·18
„ „	57·70	31·43 *h* + 28·00 *k*	31·88 *h* + 10·95 *k*	−0·47 *h* + 17·05 *k*	13·23
„ „	57·70	34·58 *h* + 24·58 *k*	30·35 *h* + 12·68 *k*	4·23 *h* + 11·90 *k*	12·86
„ „	57·72	29·08 *h* + 30·40 *k*	19·28 *h* + 23·73 *k*	9·80 *h* + 6·67 *k*	13·11
„ 20	61·72	21·30 *h* + 29·45 *k*	18·08 *h* + 16·93 *k*	3·22 *h* + 12·52 *k*	12·55
„ „	61·78	33·05 *h* + 17·85 *k*	19·45 *h* + 16·18 *k*	13·60 *h* + 1·67 *k*	12·14
„ „	61·83	26·45 *h* + 24·60 *k*	17·38 *h* + 17·78 *k*	9·07 *h* + 6·82 *k*	12·65
„ „	61·91	28·05 *h* + 23·08 *k*	19·95 *h* + 15·03 *k*	8·10 *h* + 8·05 *k*	12·86
July 4	61·02	45·53 *h* + 49·68 *k*	38·30 *h* + 41·68 *k*	7·23 *h* + 8·00 *k*	12·13
„ „	61·08	51·00 *h* + 47·03 *k*	37·80 *h* + 43·90 *k*	13·20 *h* + 3·13 *k*	−12·99

RIGHT FOOT.

Date. 1864.	Temp.	OI_1	OF	Difference of Length in Micrometer Divisions.	$[d \cdot e] - F$
June 21	61·96°	36·10 *h* + 27·45 *k*	43·88 *h* + 49·03 *k*	7·78 *h* + 21·58 *k*	+23·41
„ „	62·03	22·35 *h* + 41·60 *k*	34·33 *h* + 58·03 *k*	11·98 *h* + 16·43 *k*	22·63
„ „	62·09	31·10 *h* + 31·65 *k*	46·88 *h* + 45·28 *k*	15·78 *h* + 13·63 *k*	23·42
„ „	62·19	34·78 *h* + 28·25 *k*	46·45 *h* + 45·93 *k*	11·67 *h* + 17·68 *k*	23·39
„ 22	62·26	29·08 *h* + 33·03 *k*	45·73 *h* + 46·75 *k*	16·65 *h* + 13·72 *k*	24·18
„ „	62·35	31·38 *h* + 30·68 *k*	45·55 *h* + 47·03 *k*	14·17 *h* + 16·35 *k*	24·31
„ „	62·43	34·35 *h* + 27·83 *k*	44·98 *h* + 48·20 *k*	10·63 *h* + 20·37 *k*	24·71
„ „	62·49	24·38 *h* + 39·15 *k*	39·63 *h* + 52·75 *k*	15·25 *h* + 13·60 *k*	22·98
July 4	61·11	27·25 *h* + 25·45 *k*	45·25 *h* + 36·93 *k*	18·00 *h* + 10·48 *k*	22·67
„ „	61·29	20·50 *h* − 71·08 *k*	22·30 *h* − 43·93 *k*	1·80 *h* + 27·15 *k*	+23·10

COMPARISONS OF O_1 AND OI_1.

Date. 1864.	Temp.	O_1	OI_1	Difference of Length in Micrometer Divisions.	$O_1 - OI_1$
Feb. 18	40°·70	45·46 *h* + 45·34 *k*	31·33 *h* + 34·09 *k*	14·13 *h* + 11·25 *k*	−20·21
„ „	40·67	35·73 *h* + 52·02 *k*	33·40 *h* + 34·26 *k*	2·33 *h* + 17·76 *k*	16·02
„ „	40·74	34·52 *h* + 53·20 *k*	24·21 *h* + 43·49 *k*	10·31 *h* + 9·71 *k*	15·94
„ „	40·92	42·39 *h* + 43·58 *k*	28·53 *h* + 34·53 *k*	13·86 *h* + 9·05 *k*	18·24
„ 19	40·95	35·58 *h* + 48·66 *k*	28·86 *h* + 30·37 *k*	6·72 *h* + 18·29 *k*	19·94
„ „	40·95	30·99 *h* + 46·37 *k*	27·99 *h* + 29·11 *k*	3·00 *h* + 17·26 *k*	16·16
„ „	40·91	32·62 *h* + 46·81 *k*	28·27 *h* + 30·88 *k*	4·35 *h* + 15·93 *k*	16·17
„ „	40·82	33·75 *h* + 44·02 *k*	31·49 *h* + 25·99 *k*	2·26 *h* + 18·03 *k*	16·18
„ 20	40·33	33·36 *h* + 53·21 *k*	28·50 *h* + 36·97 *k*	4·86 *h* + 16·24 *k*	16·82
„ „	40·30	31·97 *h* + 52·63 *k*	25·18 *h* + 37·27 *k*	6·79 *h* + 15·36 *k*	17·65
May 16	62·40	99·48 *h* + 99·78 *k*	90·41 *h* + 85·69 *k*	9·07 *h* + 14·09 *k*	18·45
„ „	62·50	102·61 *h* + 96·09 *k*	88·24 *h* + 84·53 *k*	14·37 *h* + 11·56 *k*	20·65
„ „	62·58	83·16 *h* + 89·96 *k*	99·68 *h* + 100·11 *k*	16·52 *h* + 10·15 *k*	21·23
„ 17	62·64	102·48 *h* + 96·50 *k*	81·88 *h* + 92·58 *k*	20·60 *h* + 3·92 *k*	19·50
„ „	62·64	70·33 *h* + 128·80 *k*	65·38 *h* + 106·85 *k*	4·95 *h* + 21·95 *k*	21·45
„ „	62·71	90·53 *h* + 107·93 *k*	60·58 *h* + 112·98 *k*	29·95 *h* − 5·05 *k*	19·79
„ „	62·78	100·60 *h* + 99·93 *k*	79·53 *h* + 94·58 *k*	21·07 *h* + 5·35 *k*	21·02
„ „	62·88	102·65 *h* + 96·53 *k*	91·68 *h* + 82·55 *k*	10·97 *h* + 13·98 *k*	19·88
„ 18	63·04	102·73 *h* + 94·70 *k*	78·48 *h* + 92·55 *k*	24·25 *h* + 2·15 *k*	20·99
„ „	63·13	91·03 *h* + 100·05 *k*	78·63 *h* + 90·80 *k*	12·40 *h* + 9·25 *k*	17·24
„ „	63·18	89·10 *h* + 105·23 *k*	85·93 *h* + 85·58 *k*	3·17 *h* + 19·65 *k*	18·20
„ „	63·26	95·85 *h* + 99·05 *k*	84·20 *h* + 85·95 *k*	11·65 *h* + 13·10 *k*	19·71
June 4	58·35	4·40 *h* + 32·70 *k*	1·23 *h* + 7·75 *k*	3·17 *h* + 24·95 *k*	22·43
„ „	58·42	14·65 *h* + 15·60 *k*	6·08 *h* + 0·43 *k*	8·57 *h* + 15·17 *k*	18·92
„ „	58·50	20·23 *h* + 14·40 *k*	2·80 *h* + 4·60 *k*	17·43 *h* + 9·80 *k*	21·68
„ „	58·73	15·18 *h* + 14·45 *k*	1·80 *h* + 3·33 *k*	13·38 *h* + 11·12 *k*	19·51
„ 6	60·25	− 1·43 *h* + 1·65 *k*	− 9·95 *h* − 13·50 *k*	8·52 *h* + 15·15 *k*	18·86
„ „	60·42	− 6·15 *h* − 1·25 *k*	−13·35 *h* − 14·03 *k*	7·20 *h* + 12·78 *k*	15·92
„ „	60·59	− 5·98 *h* − 1·23 *k*	−16·23 *h* − 13·00 *k*	10·25 *h* + 11·77 *k*	17·54
„ „	60·84	− 2·38 *h* − 6·73 *k*	−14·53 *h* − 15·00 *k*	12·15 *h* + 8·27 *k*	16·26
„ 7	61·74	−15·03 *h* − 11·03 *k*	−12·70 *h* − 35·25 *k*	−2·33 *h* + 24·22 *k*	17·48
„ „	61·93	78·08 *h* + 92·15 *k*	64·85 *h* + 83·65 *k*	13·23 *h* + 8·50 *k*	17·30
„ „	62·06	77·95 *h* + 89·83 *k*	69·50 *h* + 77·75 *k*	8·45 *h* + 12·08 *k*	16·36
„ 23	62·37	52·03 *h* + 61·73 *k*	43·70 *h* + 49·55 *k*	8·33 *h* + 12·18 *k*	16·34
„ „	62·38	51·73 *h* + 58·40 *k*	38·35 *h* + 53·98 *k*	13·38 *h* + 4·42 *k*	14·16
„ „	62·39	44·08 *h* + 69·33 *k*	43·28 *h* + 49·18 *k*	0·80 *h* + 20·15 *k*	16·72
„ „	62·50	47·35 *h* + 64·60 *k*	45·95 *h* + 45·68 *k*	1·40 *h* + 18·92 *k*	16·21
„ „	62·56	53·98 *h* + 57·45 *k*	47·05 *h* + 44·43 *k*	6·93 *h* + 13·02 *k*	15·90
„ 24	62·22	59·75 *h* + 57·70 *k*	48·55 *h* + 48·25 *k*	11·20 *h* + 9·45 *k*	16·44
„ „	62·21	58·00 *h* + 56·78 *k*	54·05 *h* + 40·63 *k*	3·95 *h* + 16·15 *k*	−16·03

The second column in these Tables gives the mean of the eight thermometer readings at each visit, corrected for the errors of the thermometers. The third and fourth columns give the mean micrometer readings on the two bars; *h* and *k* represent the values of a division of the micrometer microscopes H and K respectively. It should also be remembered, in order perfectly to understand the result, that (as is invariably the practice) the micrometer heads are outwards, and the direction of positive measurement is towards the centre of the bar; the distance apart of the zeros of micrometers being in general greater than the length of either bar, in order that positive readings may be obtained. A departure from this rule will be observed in the comparisons of O_1 and OI_1 on June 6th; this is the result of the change of temperature which took place between the 4th and 6th, whereby the length of the bars was made to exceed the distance of the zeros of the microscopes. The values of h and k expressed in millionths of the yard are,

$$h = 0\cdot79494$$
$$k = 0\cdot79800$$

The last column contains the resulting difference of length expressed in *millionths of a yard.*

2.

In determining from the four series of comparisons on $[a\cdot e]$ and its parts the length of that space in terms of $\mathbf{Y}_{55}$ and the relative expansion of the two metals, we must form equations of condition, and solve them by the method of least squares. Let u be the excess of the expansion of one yard of $\mathbf{OI}_1$ above a yard of $\mathbf{Y}_{55}$ for one degree Fahrenheit, and let

$$\begin{aligned}[a\cdot b] &= \tfrac{1}{3}\,\mathbf{Y}_{55} + x + \tfrac{1}{3}\,u\,(t - 62) \\ [a\cdot d] &= \mathbf{Y}_{55} + y + u\,(t - 62) \\ [a\cdot e] &= \tfrac{4}{3}\,\mathbf{Y}_{55} + z + \tfrac{4}{3}\,u\,(t - 62)\end{aligned} \qquad (1)$$

Also let $\mathbf{F} = \tfrac{1}{3}\,\mathbf{Y}_{55} + \lambda + \mu\,(t - 62)$

Then
$$\begin{aligned}[a\cdot b] &= \mathbf{F} + x - \lambda + (\tfrac{1}{3}u - \mu)\,(t - 62) \\ [a\cdot d] &= \mathbf{Y}_{55} + y + u\,(t - 62) \\ [b\cdot e] &= \mathbf{Y}_{55} + z - x + u\,(t - 62) \\ [d\cdot e] &= \mathbf{F} + z - y - \lambda + (\tfrac{1}{3}u - \mu)\,(t - 62)\end{aligned}$$

The values we obtained for λ and μ were—

$$\lambda = -0\cdot36 \quad ; \quad \mu = 0\cdot0066$$

There will be 63 equations, as follows :—

$$\begin{aligned}
x - 1\cdot44\,u + 12\cdot57 &= 0 \\
x - 1\cdot43\,u + 13\cdot62 &= 0 \\
x - 1\cdot43\,u + 13\cdot25 &= 0 \\
x - 1\cdot43\,u + 13\cdot50 &= 0 \\
x - 0\cdot09\,u + 12\cdot91 &= 0 \\
x - 0\cdot07\,u + 12\cdot50 &= 0 \\
x - 0\cdot06\,u + 13\cdot01 &= 0 \\
x - 0\cdot03\,u + 13\cdot22 &= 0 \\
x - 0\cdot33\,u + 12\cdot50 &= 0 \\
x - 0\cdot31\,u + 13\cdot36 &= 0 \\
y - 23\cdot39\,u + 44\cdot27 &= 0 \\
y - 23\cdot21\,u + 44\cdot19 &= 0 \\
y - 23\cdot17\,u + 44\cdot38 &= 0 \\
y - 23\cdot49\,u + 45\cdot39 &= 0 \\
y - 23\cdot52\,u + 44\cdot32 &= 0 \\
y + 1\cdot58\,u + 46\cdot83 &= 0 \\
y + 1\cdot67\,u + 46\cdot47 &= 0 \\
y + 1\cdot74\,u + 46\cdot92 &= 0 \\
y + 1\cdot74\,u + 47\cdot17 &= 0 \\
y + 0\cdot04\,u + 47\cdot95 &= 0 \\
y + 0\cdot10\,u + 45\cdot53 &= 0 \\
y + 0\cdot11\,u + 47\cdot42 &= 0
\end{aligned}$$

$$y + 0.15\,u + 47.02 = 0$$
$$y + 1.87\,u + 48.38 = 0$$
$$y + 1.97\,u + 45.73 = 0$$
$$y + 2.05\,u + 47.23 = 0$$
$$y + 2.16\,u + 46.80 = 0$$
$$y - 0.74\,u + 46.57 = 0$$
$$y - 0.72\,u + 44.78 = 0$$
$$y - 0.70\,u + 47.28 = 0$$
$$y - 0.45\,u + 47.96 = 0$$
$$-x + z - 23.33\,u + 10.24 = 0$$
$$-x + z - 23.72\,u + 10.90 = 0$$
$$-x + z - 23.80\,u + 10.38 = 0$$
$$-x + z - 23.75\,u + 10.22 = 0$$
$$-x + z - 23.73\,u + 10.54 = 0$$
$$-x + z - 23.96\,u + 9.95 = 0$$
$$-x + z - 0.16\,u + 7.66 = 0$$
$$-x + z - 0.16\,u + 11.00 = 0$$
$$-x + z - 1.11\,u + 9.30 = 0$$
$$-x + z - 1.14\,u + 8.73 = 0$$
$$x + z - 1.15\,u + 8.98 = 0$$
$$-x + z - 1.72\,u + 10.13 = 0$$
$$-x + z - 1.69\,u + 8.93 = 0$$
$$-x + z - 1.69\,u + 9.89 = 0$$
$$-x + z + 2.22\,u + 11.50 = 0$$
$$-x + z - 0.45\,u + 11.31 = 0$$
$$-x + z - 0.50\,u + 10.34 = 0$$
$$-x + z - 0.51\,u + 11.36 = 0$$
$$-x + z - 0.21\,u + 12.31 = 0$$
$$-x + z - 0.68\,u + 8.96 = 0$$
$$-x + z - 0.64\,u + 10.19 = 0$$
$$-x + z - 0.56\,u + 9.87 = 0$$
$$-y + z - 0.01\,u - 23.05 = 0$$
$$-y + z + 0.01\,u - 22.27 = 0$$
$$-y + z + 0.03\,u - 23.06 = 0$$
$$-y + z + 0.06\,u - 23.03 = 0$$
$$-y + z + 0.09\,u - 23.82 = 0$$
$$-y + z + 0.12\,u - 23.95 = 0$$
$$-y + z + 0.14\,u - 24.35 = 0$$
$$-y + z + 0.16\,u - 22.62 = 0$$
$$-y + z - 0.30\,u - 22.30 = 0$$
$$-y + z - 0.24\,u - 22.74 = 0$$

Forming the four final equations according to the method of least squares, we find—

$$\begin{array}{rrrrrl} +\,32.00\,x & & -\,22.00\,z & +\,145.82\,u & -\,92.25 & = 0 \\ & +\,31.00\,y & -\,10.00\,z & -\,104.27\,u & +\,1203.79 & = 0 \\ -\,22.00\,x & -\,10.00\,y & +\,32.00\,z & -\,152.38\,u & -\,8.50 & = 0 \\ +\,145.82\,x & -\,104.27\,y & -\,152.38\,z & +\,6159.735\,u & -\,6265.945 & = 0 \end{array} \qquad (2)$$

If we write A B C D for the absolute terms of these four equations, and then eliminate $x\ y\ z\ u$, there results—

$$\begin{array}{l} x + .06589108\,A + .01597077\,B + .05004595\,C - .00005146\,D = 0 \\ y + .01597077\,A + .04619088\,B + .03098788\,C + .00117041\,D = 0 \\ z + .05004595\,A + .03098788\,B + .08183691\,C + .00136430\,D = 0 \\ u - .00005146\,A + .00117041\,B + .00136430\,C + .00021712\,D = 0 \end{array} \qquad (3)$$

The actual values of $x\ y\ z\ u$ and the reciprocals of the weights of the determinations are consequently

$$x = -13{\cdot}044 \qquad \textit{Reciprocal of weight} = {\cdot}06589108$$
$$y = -46{\cdot}534 \qquad \text{,,} \qquad {\cdot}04619088$$
$$z = -23{\cdot}442 \qquad \text{,,} \qquad {\cdot}08183691$$
$$u = -0{\cdot}0416 \qquad \text{,,} \qquad {\cdot}00021712$$

If now we substitute in the equations of condition these values of $x\ y\ z$ and u we get the errors of the individual comparisons, which we shall put together in the following Table in the order of dates.

First Series.		Second Series.		Third Series.		Fourth Series.	
Date.	Error.	Date.	Error.	Date.	Error.	Date.	Error.
			Left Foot : [a·b].				
June 2	− 0·41	June 20	− 0·13	July 4	− 0·53		
,, ,,	+ 0·64	,, ,,	− 0·54	,, ,,	+ 0·33		
,, ,,	+ 0·27	,, ,,	− 0·03				
,, ,,	+ 0·52	,, ,,	+ 0·18				
			Left Centre Yard : [a·d].				
Feb. 22	− 1·29	May 23	+ 0·23	June 9	+ 1·77	June 29	+ 0·07
,, ,,	− 1·37	,, ,,	− 0·13	,, ,,	− 0·88	,, ,,	− 1·72
,, ,,	− 1·19	,, ,,	+ 0·32	,, ,,	+ 0·62	,, ,,	+ 0·78
,, 23	− 0·16	,, ,,	+ 0·57	,, ,,	+ 0·18	,, ,,	+ 1·45
,, ,,	− 1·23	,, 25	+ 1·42				
		,, ,,	− 1·00				
		,, ,,	+ 0·89				
		,, ,,	+ 0·48				
			Right Centre Yard : [b·e].				
Feb. 23	+ 0·81	May 26	− 2·73	June 10	+ 1·01	June 29	+ 1·92
,, 24	+ 1·49	,, ,,	+ 0·61	,, 17	+ 0·93	,, 30	− 1·41
,, ,,	+ 0·97	,, 27	− 1·05	,, ,,	− 0·04	,, ,,	− 0·18
,, ,,	+ 0·81	,, ,,	− 1·62	,, ,,	+ 0·98	,, ,,	− 0·51
,, ,,	+ 1·13	,, ,,	− 1·37				
,, 25	+ 0·55	,, 28	− 0·20				
		,, ,,	− 1·40				
		,, ,,	− 0·44				
			Right Foot : [d·e].				
June 21	+ 0·04	July 4	+ 0·80				
,, ,,	+ 0·82	,, ,,	+ 0·36				
,, ,,	+ 0·03						
,, ,,	+ 0·06						
,, 22	− 0·73						
,, ,,	− 0·86						
,, ,,	− 1·27						
,, ,,	+ 0·46						

The only point that calls for remark in the system of errors here shown is the difference of sign of the errors in February on the two centre yards, as though the rate of expansion that suited the one did not suit the other. As the two yards overlap and have two feet of length in common, it is difficult to conceive any cause for this discrepancy.

The sum of the squares of the sixty-three errors is 57·7166. Therefore the probable error of a single comparison is—

$$\pm 0.674 \sqrt{\frac{57{\cdot}7166}{63-4}} = \pm 0{\cdot}667 \qquad (4)$$

Making use of the weights we have obtained for z and u, we get for the

$$\textit{Probable error of } z = \pm 0{\cdot}667 \sqrt{{\cdot}08184} = \pm 0{\cdot}191 \qquad (5)$$
$$\text{,,} \qquad \textit{of } u = \pm 0{\cdot}667 \sqrt{{\cdot}000217} = \pm 0{\cdot}0098$$

For the space $[a \cdot e]$ we have then the length—

$$[a \cdot e] = \tfrac{4}{3}\, \mathbf{Y}_{55} - 23{\cdot}44 - 0{\cdot}0555\,(t - 62) \qquad (6)$$

when the coefficient of $(t - 62)$ is $\frac{4}{3}u$.

But we have yet to consider the probable error to which this result is subject in consideration of the probable error of the foot **OF** with which the spaces $[a \cdot b]$ and $[d \cdot e]$ were compared.

In the first place it is to be noted that these comparisons were made very close to the normal temperature of 62°, consequently any slight uncertainty in the difference of expansion of **OF** and $\mathbf{Y}_{55}$ (which has been well determined) will not affect our results. But if in our equations of condition we had taken as the length of **F** at 62°, $\frac{1}{3}\mathbf{Y}_{55} - 0{\cdot}36 + n$, and if we had retained this symbol to the end of the calculation, we should have obtained, as is easily verified—

$$x = -\ 13{\cdot}044 + n$$
$$y = -\ 46{\cdot}534$$
$$z = -\ 23{\cdot}442 + n$$
$$u = -\ \ 0{\cdot}0416$$

On substituting these values in our equations of condition we should obtain the same system of errors as we have already exhibited, and hence to take fully into account the uncertainty in the value of **F** in our resulting value of z we must add to the square of the probable error of z the square of the probable error of the assumed length of **OF** at 62°. This last probable error is (see page 77) ± 0·108. Therefore the probable error of $[a \cdot e]$ at 62° is—

$$\sqrt{({\cdot}191)^2 + ({\cdot}108)^2} = \pm 0{\cdot}219,$$

and consequently at the temperature of 62°

$$[a \cdot e] = \tfrac{4}{3}\, \mathbf{Y}_{55} - 23{\cdot}44 \pm 0{\cdot}219. \qquad (7)$$

3.

We have defined u to be the excess of the expansion of one yard of $\mathbf{OI}_1$ above that of $\mathbf{Y}_{55}$ for one degree Fahrenheit. The value of u being negative we have this result,—that the expansion of the yard $\mathbf{Y}_{55}$ *exceeds* the expansion of three feet of $\mathbf{OI}_1$ by 0·0416 ± ·0098 for 1° Fahrenheit.

Now by direct experiments the following result has been found for $\mathbf{OI}_1$ (see page 79):

$$\textit{Expansion for } 1^\circ \textit{ Fahrenheit} = 21{\cdot}055 \pm {\cdot}089\,;$$

therefore the expansion of *three feet* of this bar is

$$6{\cdot}3165 \pm {\cdot}0267.$$

If from this we inferred the expansion of $\mathbf{Y}_{55}$ we should find—

$$\textit{Expansion of } \mathbf{Y}_{55} = 6{\cdot}358 \pm {\cdot}0284. \qquad (8)$$

4.

We have seen that the excess of the expansion of one yard of **OI**$_1$ above that of **Y**$_{55}$ for one degree Fahrenheit is

$$-0{\cdot}0416.$$

If by means of this we reduce the comparisons of the right and left yards of **OI**$_1$ to the temperature of 62°, we get the following values :—

Left Yard at 62°.		Right Yard at 62°.	
Value.	Errors.	Value.	Errors.
Y$_{55}$ + 54·80	+0·05	**Y**$_{55}$ − 9·98	+0·25
53·42	−1·33	9·82	+0·41
53·36	−1·39	9·97	+0·26
55·13	+0·38	11·56	−1·33
54·88	+0·13	8·64	+1·59
54·38	−0·37	9·36	+0·87
53·97	−0·78	9·68	+0·55
55·55	+0·80	9·27	+0·96
54·88	+0·13	10·13	+0·10
55·86	+1·11	8·39	+1·84
54·45	−0·30	9·48	+0·75
56·21	+1·46	9·18	+1·05
54·09	−0·66	11·54	−1·31
55·22	+0·47	10·34	−0·11
55·53	+0·78	11·45	−1·22
55·39	+0·64	11·00	−0·77
54·61	−0·14	11·08	−0·85
52·92	−1·83	10·42	−0·19
54·99	+0·24	11·63	−1·40
Y$_{55}$ + 55·38	+0·63	**Y**$_{55}$ − 11·75	−1·52

Taking the means of the first and third columns, we get

$$[\alpha \cdot a] = \mathbf{Y}_{55} + 54{\cdot}75 \text{ at } 62° \textit{ Fahrenheit.} \qquad (9)$$
$$[e \cdot \epsilon] = \mathbf{Y}_{55} - 10{\cdot}23 \quad \text{,,} \quad \text{,,}$$

The sum of the squares of the errors in the left yard is 14·218, hence the probable error of a single comparison is

$$\pm {\cdot}674\sqrt{\frac{14{\cdot}218}{20-1}} = \pm {\cdot}583, \qquad (10)$$

and the probable error of the mean of the twenty comparisons

$$\pm \frac{{\cdot}583}{\sqrt{20}} = \pm 0{\cdot}130. \qquad (11)$$

The sum of the squares of the errors of comparisons in the right yard is 20·475, hence the probable error of a single comparison is

$$\pm {\cdot}674\sqrt{\frac{20{\cdot}475}{20-1}} = \pm 0{\cdot}700, \qquad (12)$$

and the probable error of the mean of the twenty comparisons

$$\pm \frac{0{\cdot}700}{\sqrt{20}} = \pm 0{\cdot}156. \qquad (13)$$

5.

It remains now to bring together the results at which we have arrived, in order to ascertain the whole length $[\alpha \cdot \epsilon]$ of $\mathbf{OI}_1$ in terms of $\mathbf{Y}_{55}$ at 62°.

We have found

$$[\alpha \cdot a] = \mathbf{Y}_{55} + 54.75 \pm 0.130$$
$$[a \cdot e] = \tfrac{4}{3}\mathbf{Y}_{55} - 23.44 \pm 0.219$$
$$[e \cdot \epsilon] = \mathbf{Y}_{55} - 10.23 \pm 0.156$$

and adding these together, we have finally

$$\mathbf{OI}_1 = \tfrac{10}{3}\mathbf{Y}_{55} + 21.08 \pm 0.299. \qquad (14)$$

6.

For the difference of length of $\mathbf{OI}_1$ and $\mathbf{O}_1$ at 62° Fahrenheit and the difference of their expansions we have ten comparisons at 40° and thirty comparisons near 62°. Let y be the excess of expansion of $\mathbf{O}_1$ above $\mathbf{OI}_1$ for one degree of Fahrenheit, and let x be the excess of the length of $\mathbf{O}_1$ above $\mathbf{OI}_1$ at 62°; then in general

$$\mathbf{O}_1 = \mathbf{OI}_1 + x + y\,(t - 62).$$

The equations resulting from the forty comparisons are as follows:—

$x - 21.30\,y + 20.21 = 0$
$x - 21.33\,y + 16.02 = 0$
$x - 21.26\,y + 15.94 = 0$
$x - 21.08\,y + 18.24 = 0$
$x - 21.05\,y + 19.94 = 0$
$x - 21.05\,y + 16.16 = 0$
$x - 21.09\,y + 16.17 = 0$
$x - 21.18\,y + 16.18 = 0$
$x - 21.67\,y + 16.82 = 0$
$x - 21.70\,y + 17.65 = 0$
$x + 0.40\,y + 18.45 = 0$
$x + 0.50\,y + 20.65 = 0$
$x + 0.58\,y + 21.23 = 0$
$x + 0.64\,y + 19.50 = 0$
$x + 0.64\,y + 21.45 = 0$
$x + 0.71\,y + 19.79 = 0$
$x + 0.78\,y + 21.02 = 0$
$x + 0.88\,y + 19.88 = 0$
$x + 1.04\,y + 20.99 = 0$
$x + 1.13\,y + 17.24 = 0$
$x + 1.18\,y + 18.20 = 0$
$x + 1.26\,y + 19.71 = 0$
$x - 3.65\,y + 22.43 = 0$
$x - 3.58\,y + 18.92 = 0$
$x - 3.50\,y + 21.68 = 0$
$x - 3.27\,y + 19.51 = 0$
$x - 1.75\,y + 18.86 = 0$
$x - 1.58\,y + 15.92 = 0$
$x - 1.41\,y + 17.54 = 0$
$x - 1.16\,y + 16.26 = 0$
$x - 0.26\,y + 17.48 = 0$
$x - 0.07\,y + 17.30 = 0$
$x + 0.06\,y + 16.36 = 0$
$x + 0.37\,y + 16.34 = 0$
$x + 0.38\,y + 14.16 = 0$
$x + 0.39\,y + 16.72 = 0$
$x + 0.50\,y + 16.21 = 0$
$x + 0.56\,y + 15.90 = 0$
$x + 0.22\,y + 16.44 = 0$
$x + 0.21\,y + 16.03 = 0$

From these we obtain

$$\begin{aligned} 40\,x - 220{\cdot}51\,y + 725{\cdot}50 &= 0 \\ -\,220{\cdot}51\,x + 4593{\cdot}01\,y - 3848{\cdot}76 &= 0 \end{aligned} \tag{15}$$

Putting A B for the absolute terms of these equations,

$$\begin{aligned} 0 &= x + {\cdot}0339982\,\mathrm{A} + {\cdot}0016322\,\mathrm{B} \\ 0 &= y + {\cdot}0016322\,\mathrm{A} + {\cdot}0002961\,\mathrm{B} \end{aligned} \tag{16}$$

The actual values of x and y are—

$$\begin{aligned} x &= -\,18{\cdot}384 \\ y &= -\,0{\cdot}0446, \end{aligned} \tag{17}$$

and consequently the length of $\mathbf{O}_1$ is expressed in terms of $\mathbf{OI}_1$ by the formula—

$$\mathbf{O}_1 = \mathbf{OI}_1 - 18{\cdot}384 - 0{\cdot}0446\,(t - 62). \tag{18}$$

If we substitute the above values of x and y in the equations we obtain the following system of errors:—

First Series.		Second Series.		Third Series.		Fourth Series.	
Date.	Error.	Date.	Error.	Date.	Error.	Date.	Error.
Feb. 18	+ 2·78	May 16	+ 0·05	June 4	+ 4·21	June 23	− 2·06
" "	− 1·41	" "	+ 2·24	" "	+ 0·70	" "	− 4·24
" "	− 1·50	" "	+ 2·82	" "	+ 3·45	" "	− 1·68
" "	+ 0·80	" 17	+ 1·09	" "	+ 1·27	" "	− 2·20
" 19	+ 2·50	" "	+ 3·04	" 6	+ 0·55	" "	− 2·51
" "	− 1·28	" "	+ 1·37	" "	− 2·39	" 24	− 1·95
" "	− 1·27	" "	+ 2·60	" "	− 0·78	" "	− 2·36
" "	− 1·26	" "	+ 1·46	" "	− 2·07		
" 20	− 0·60	" 18	+ 2·56	" 7	− 0·89		
" "	+ 0·23	" "	− 1·19	" "	− 1·08		
		" "	− 0·24	" "	− 2·03		
		" "	+ 1·27				

There are here two large errors of equal magnitude and opposite, viz., + 4·21 on June 4th and − 4·24 June 23d. There is nothing in the observations to throw any doubt whatever on these determinations. If we compare them with adjacent ones they assume the appearance of accidental errors.

Upon the whole the errors have not entirely the appearance of accidental errors. On some days there is a preponderance of positive errors, and on others a preponderance of negative errors. This is not any effect of personal error, as two different observers and sometimes three have made the comparisons on each day. It is to be remembered that one observer only is employed in a single comparison.

Another possible cause which presents itself is that one bar might be at a slightly different temperature from the other; but this is not borne out by an examination of the readings of the thermometer in the two bars. Take, for instance, the observations on May 17th and June 23d. It will be remembered that each bar carries two thermometers: those in $\mathbf{O}_1$ were marked Simms, No. 1 and No. 2, those in $\mathbf{OI}_1$ were Casella, 3461 and 3462. The error of the mean of 1 and 2 about 62° is + 0°·07, and the error of the mean of 3461 and 3462 at the same temperature is + 0°·19.

The temperatures of the bars on the days in question are:—

Hour.	Temperature May 17.		Temperature June 23.	
	OI_1	O_1	OI_1	O_1
	°	°	°	°
9 A.M.	62·80	62·74	62·53	62·46
12 „	62·80	62·74	62·55	62·48
2 P.M.	62·87	62·81	62·56	62·49
5 „	62·95	62·87	62·64	62·61
9 „	63·03	62·99	62·74	62·64
Mean	62·89	62·83	62·60	62·54
Corr.	—·19	—·07	—·19	—·07
	62·70	62·76	62·41	62·47

From this it would appear that on each of these days the temperature of O_1 exceeded by 0°·06 that of OI_1. We should therefore expect to find no difference in the determinations on these days, *so far as can be ascertained* by the indication of the thermometers in the bars.

In considering the magnitude of some of the errors shown in this Table, it must be borne in mind that the diameters of the dots on O_1 are each one hundred divisions. With this in view the errors are small.

The sum of the squares of the forty errors is 162·19; hence the probable error of a single comparison is

$$\pm \cdot 674 \sqrt{\frac{162\cdot 19}{40-2}} = \pm 1\cdot 391, \tag{19}$$

and the probable error of x is

$$= \pm 1\cdot 391 \sqrt{\cdot 03400} = \pm 0\cdot 256, \tag{20}$$

and the probable error of y is

$$= \pm 1\cdot 391 \sqrt{\cdot 000296} = \pm 0\cdot 0239. \tag{21}$$

Finally, at 62°, we have for the length of O_1

$$O_1 = OI_1 - 18\cdot 38 \pm 0\cdot 26. \tag{22}$$

7.

We can now express the length of O_1 at 62° in terms of Y_{55}, for we have found

$$O_1 = OI_1 - 18\cdot 38 \pm 0\cdot 256$$
$$OI_1 = \tfrac{10}{3} Y_{55} + 21\cdot 08 \pm 0\cdot 299 ;$$

consequently, adding these equations, we get for the length of the Ordnance Survey Standard:—

$$O_1 = (3\cdot 33333603 \pm \cdot 00000039)\, Y_{55} \tag{23}$$

and the logarithm by which all distances which have been expressed in terms of the Ordnance Standard $\mathbf{O}_1$ will have to be multiplied is

0·00000035.

This result agrees very satisfactorily with that obtained with the old apparatus some years since from the Australian Standard Bars, which will be found in a subsequent section.

The probable error of the above determination of the length of $\mathbf{O}_1$, expressed as a fraction of the whole length 10 feet, is

$$\frac{1}{8{,}410{,}000}.$$

8.

From the comparisons of $\mathbf{O}_1$ and $\mathbf{OI}_1$ just detailed, we have seen that the expansion of $\mathbf{O}_1$ is *less* than the expansion of $\mathbf{OI}_1$ by

$$0·0446 \pm 0·0239.$$

But the absolute expansion of $\mathbf{OI}_1$ we have found by direct experiment to be 21·055 ± ·089; therefore the inferred expansion of $\mathbf{O}_1$

$$= 21·010 \pm 0·092,$$

which is equivalent to ·000006303 on unity for one degree Fahrenheit.

The rate of expansion deduced from comparisons with the Compensation Bars during the measurement of the base line on Salisbury Plain was ·00000637 on unity. This agreement is tolerably satisfactory.

9.

There is a considerable difference between the average magnitude of errors of comparison of a foot, a yard, and ten feet, as exhibited in the results of the observation we have just been discussing. If, disregarding the signs of the errors, we simply take the arithmetic mean in the seven different operations, we get the following average errors:—

Left foot	0·36
Right foot	0·54
Left yard	0·68
Left centre yard	0·85
Right centre yard	1·01
Right yard	0·87
Ten feet	1·75

Hence the average errors of a comparison on a foot, a yard, and ten feet are 0·45; 0·85; 1·75; or as

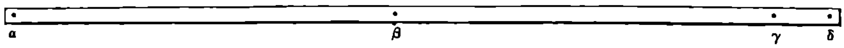
$$1·00 : 1·89 : 3·89.$$

These are not in the proportion of the corresponding lengths, but are not far from being in proportion to the square roots of the lengths. If, for instance, the errors were 0·53, 0·92, 1·68, they would be strictly as the square roots of the lengths, and these are not far, any of them, from the actual quantities.

VIII.

DETERMINATION OF THE LENGTH OF THE ORDNANCE TOISE.

There are, on the upper surface of this bar, as has already been described, four points α β γ δ



of which α and β are one yard apart, β and γ also one yard apart, and γ and δ 4·74 inches apart; so that the whole length, or $[\alpha \cdot \delta] = 76{\cdot}74$ inches approximately. Each of the spaces $[\alpha \cdot \beta]$ and $[\beta \cdot \gamma]$ has been directly compared with the copy No. 55 of the Standard Yard, and the space $[\gamma \cdot \delta]$ with the space $[f \cdot \tau]$ on the bar **OF.**

1.

The comparisons of the yard $[\beta \cdot \gamma]$ of the Ordnance Toise with Standard Yard No. 55 extend over twelve days. Eight days in October at an average temperature of about 59°, and four days in January 1864 at a temperature 41°. The comparisons of the yard $[\alpha \cdot \beta]$ extend over ten days in October 1863 at a temperature averaging 60°, and four days in January 1864 at a temperature averaging 38°. In all, thirty comparisons of each yard; if here by "comparison" we mean the result of a single visit.

The observations were generally made three times in the day, at hours as far apart as practicable. The bars were observed in the following order $\mathbf{Y}_{55}$, **OT**; **OT**, $\mathbf{Y}_{55}$: $\mathbf{Y}_{55}$, **OT**; (three readings of each microscope being taken;) or **OT**, $\mathbf{Y}_{55}$; $\mathbf{Y}_{55}$, **OT**: **OT**, $\mathbf{Y}_{55}$; the bars *left* under the microscopes being thus alternately one and the other. Each visit supplies, therefore, three independent comparisons, in all thirty-six micrometer readings.

Each bar has two thermometers, which are read the first thing and the last in each visit. In this series the mean increase of temperature as shown by the comparison of the thermometer readings at the commencement and close of each visit is 0°·02, being the effect of the warmth of the observer's body and of the candles.

The following copy of a page of the observation book shows the exact method of recording the observations :—

YARD [β·γ]

Date Observer.	Bar.	Micr. Readings.		Thermometers.	
		H	K	Y_{55}	OT
Jan. 19, 9.15 A.M.	Y_{55}	19·5 19·9 19·4	12·7 13·6 14·2	41·37 41·35	41·30 41·30
	OT	4·4 3·6 4·3	13·4 14·3 14·5		
	OT	4·4 4·2 4·5	13·0 13·5 13·5		
	Y_{55}	13·5 13·6 14·0	19·0 18·6 19·3		
	Y_{55}	13·4 13·4 13·6	19·7 19·5 18·7		
	OT	13·5 13·6 13·8	3·5 3·3 3·5	41·38 41·39	41·31 41·32

The mean of the nine readings of H and K on Y_{55} are

15·59 . . . . 17·26

The mean of the nine readings of H and K on **OT** are

7·37 . . . . 10·28

consequently if h and k be the values of one division in H and K respectively

$$(15{\cdot}59\,h + 17{\cdot}26\,k) - (7{\cdot}37\,h + 10{\cdot}28\,k) = 8{\cdot}22\,h + 6{\cdot}98\,k$$

will be the difference of length resulting from this visit, and is virtually the mean of three comparisons.

The microscope H (invariably in all comparisons on the left) has its micrometer head to the left; K (invariably in all comparisons on the right) has its micrometer head to the right. The positive direction of measurement in either microscope (increasing readings) is towards the centre of the bar.

which to deduce x' and y', and as many for $x_{,}$ and $y_{,}$. These last equations, viz., in $x_{,}$ $y_{,}$ being treated according to the method of least squares, resolve themselves into the following:

$$30\,x_{,} - 273{\cdot}60\,y_{,} - 194{\cdot}40 = 0$$
$$-273{\cdot}60\,x_{,} + 4739{\cdot}51\,y_{,} + 2820{\cdot}58 = 0 \qquad (1)$$

which give

$$x_{,} = +2{\cdot}223$$
$$y_{,} = -0{\cdot}4668 \qquad (2)$$

The comparisons of the right yard afford the following:

$$30\,x' - 269{\cdot}45\,y' - 304{\cdot}57 = 0$$
$$-269{\cdot}45\,x' + 5637{\cdot}54\,y' + 4100{\cdot}94 = 0 \qquad (3)$$

where x' is the excess of the length of the right yard above $\mathbf{Y}_{55}$ at 62°, and y' the rate of relative expansion. The values resulting from these equations are

$$x' = +6{\cdot}341$$
$$y' = -0{\cdot}4244 \qquad (4)$$

From this it would appear as though one half of the bar **OT** had a slightly different rate of expansion from the other half. But this may well be attributed to errors in the operation, as the difference is small. If we assume, as is indeed necessary, that the rate of expansion is the same for the right yard as for the left, then the sixty comparisons must be combined in one series of equations containing three unknown quantities $x_{,}$ x' and y.

The resulting equations are as follows:—

$$30\,x_{,} - 273{\cdot}60\,y - 194{\cdot}40 = 0$$
$$30\,x' - 269{\cdot}45\,y - 304{\cdot}57 = 0 \qquad (5)$$
$$-273{\cdot}60\,x_{,} - 269{\cdot}45\,x' + 10377{\cdot}05\,y + 6921{\cdot}52 = 0$$

If we write A B C for the absolute terms of these equations we have by elimination

$$x_{,} + 0{\cdot}0485620\,A + 0{\cdot}0149977\,B + 0{\cdot}0016698\,C = 0$$
$$x' + 0{\cdot}0149977\,A + 0{\cdot}0481035\,B + 0{\cdot}0016445\,C = 0 \qquad (6)$$
$$y + 0{\cdot}0016698\,A + 0{\cdot}0016445\,B + 0{\cdot}0001831\,C = 0$$

While the numerical values are found to be

$$x_{,} = +2{\cdot}451$$
$$x' = +6{\cdot}184 \qquad (7)$$
$$y = -0{\cdot}4418$$

If from equations (6) we write out the expression for the reciprocal of the weight of $\alpha x_{,} + \beta x' + \gamma y$ and then make $\alpha = 1$, $\beta = 1$, we get the reciprocal of the weight of $x_{,} + x' + \gamma y$

$$\cdot12666 + \cdot00663\,\gamma + \cdot000183\,\gamma^2 \qquad (8)$$

Now the length of the two yards taken together is at the temperature $62° + f$ Fahrenheit

$$x_{,} + x' + 2fy$$

in excess of 2 Y_{55} at the same temperature; and the weight of the determination at that temperature is the reciprocal of

$$\cdot 12666 + \cdot 01326 f + \cdot 000732 f^2 \qquad (9)$$

The following Table contains in contrast the observed differences of length, and the differences computed from the formulæ $x_{,} + (t - 62) y$, $x' + (t - 62) y$.

COMPARISON OF YARDS OF ORDNANCE TOISE WITH THE STANDARD YARD No. 55.

$[\beta \cdot \gamma] - Y_{55}$			$[\alpha \cdot \beta] - Y_{55}$		
Observed.	Computed.	Error.	Observed.	Computed.	Error.
2·90	2·52	+ 0·38	6·66	6·63	+ 0·03
4·07	3·41	+ 0·66	6·41	7·18	− 0·77
3·19	3·33	− 0·14	8·32	7·20	+ 1·12
2·74	3·32	− 0·58	7·68	7·26	+ 0·42
3·26	3·61	− 0·35	8·63	7·66	+ 0·97
2·79	3·54	− 0·75	5·60	6·18	− 0·58
4·31	3·47	+ 0·84	5·67	6·69	− 1·02
3·33	3·40	− 0·07	7·13	6·68	+ 0·45
4·52	3·30	+ 1·22	5·85	6·68	− 0·83
3·22	3·25	− 0·03	6·70	6·68	+ 0·02
3·19	3·95	− 0·76	7·22	6·13	+ 1·09
3·31	3·95	− 0·64	7·92	6·51	+ 1·41
3·23	3·95	− 0·72	6·24	6·53	− 0·29
2·92	4·24	− 1·32	7·70	6·56	+ 1·14
4·61	4·23	+ 0·38	6·79	7·35	− 0·56
3·44	4·23	− 0·79	7·79	7·25	+ 0·54
3·26	4·51	− 1·25	5·97	7·22	− 1·25
4·73	4·50	+ 0·23	6·88	7·39	− 0·51
4·65	4·48	+ 0·17	7·75	7·36	+ 0·39
4·83	4·77	+ 0·06	8·34	7·37	+ 0·97
11·72	12·09	− 0·37	18·88	17·23	+ 1·65
11·95	12·05	− 0·10	17·40	16·97	+ 0·43
12·18	12·10	+ 0·08	15·09	16·88	− 1·79
12·79	12·01	+ 0·78	15·95	16·79	− 0·84
12·62	11·95	+ 0·67	15·79	16·65	− 0·86
12·03	11·82	+ 0·21	16·48	16·49	− 0·01
12·09	11·72	+ 0·37	16·38	16·38	+ 0·00
11·86	11·63	+ 0·23	15·68	16·27	− 0·59
12·10	11·58	+ 0·52	15·89	16·17	− 0·28
12·56	11·47	+ 1·09	15·78	16·11	− 0·33

The sum of the squares of the differences or errors is 33·75, consequently the probable error of a single comparison is

$$\pm \cdot 674 \sqrt{\frac{33 \cdot 75}{60 - 3}} = \pm \cdot 518 \qquad (10)$$

The probable error of $x_{,} + x'$ is consequently

$$\pm \cdot 518 \sqrt{\cdot 12666} = \pm 0 \cdot 184 \qquad (11)$$

The following Tables contain the result of the comparisons of the yards [α·β] and [β·γ] of the Ordnance Toise with Y_{55}: each line is the result of one visit.

COMPARISON OF YARD [α·β] OF ORDNANCE TOISE WITH THE STANDARD YARD.

Date.	Temp.	OT	Y_{55}	Difference of Length in Micrometer Divisions.	$[\alpha\cdot\beta]-Y_{55}$
1863.	°				
Oct. 5	60·98	$+36{\cdot}96h - 28{\cdot}41k$	$+22{\cdot}79h - 5{\cdot}96k$	$-14{\cdot}17h + 22{\cdot}45k$	+ 6·66
„ 6	59·73	$+70{\cdot}96h - 55{\cdot}42k$	$+0{\cdot}31h + 22{\cdot}99k$	$-70{\cdot}65h + 78{\cdot}41k$	6·41
„ „	59·68	$+67{\cdot}32h - 50{\cdot}92k$	$-1{\cdot}10h + 27{\cdot}66k$	$-68{\cdot}42h + 78{\cdot}58k$	8·32
„ „	59·56	$+10{\cdot}04h + 6{\cdot}09k$	$-10{\cdot}96h + 36{\cdot}63k$	$-21{\cdot}00h + 30{\cdot}54k$	7·68
„ 7	58·66	$+86{\cdot}72h - 65{\cdot}48k$	$-18{\cdot}78h + 50{\cdot}43k$	$-105{\cdot}50h + 115{\cdot}91k$	8·63
„ 8	62·01	$-17{\cdot}48h + 16{\cdot}86k$	$-57{\cdot}48h + 63{\cdot}72k$	$-40{\cdot}00h + 46{\cdot}86k$	5·60
„ 9	60·84	$-62{\cdot}88h + 73{\cdot}76k$	$-99{\cdot}40h + 117{\cdot}24k$	$-36{\cdot}52h + 43{\cdot}48k$	5·67
„ „	60·87	$-76{\cdot}58h + 87{\cdot}87k$	$-113{\cdot}07h + 133{\cdot}16k$	$-36{\cdot}49h + 45{\cdot}29k$	7·13
„ „	60·87	$-70{\cdot}96h + 81{\cdot}43k$	$-120{\cdot}79h + 138{\cdot}39k$	$-49{\cdot}83h + 56{\cdot}96k$	5·85
„ 10	60·87	$-92{\cdot}14h + 102{\cdot}63k$	$-121{\cdot}36h + 140{\cdot}14k$	$-29{\cdot}22h + 37{\cdot}51k$	6·70
„ 23	62·11	$-16{\cdot}07h + 16{\cdot}29k$	$-28{\cdot}59h + 37{\cdot}80k$	$-12{\cdot}52h + 21{\cdot}51k$	7·22
„ 24	61·26	$-16{\cdot}03h + 15{\cdot}29k$	$-14{\cdot}09h + 23{\cdot}28k$	$+1{\cdot}94h + 7{\cdot}99k$	7·92
„ „	61·20	$-4{\cdot}31h + 5{\cdot}02k$	$-6{\cdot}54h + 15{\cdot}06k$	$-2{\cdot}23h + 10{\cdot}04k$	6·24
„ „	61·14	$+12{\cdot}92h - 12{\cdot}44k$	$+13{\cdot}33h - 3{\cdot}20k$	$+0{\cdot}41h + 9{\cdot}24k$	7·70
„ 26	59·36	$+3{\cdot}71h + 1{\cdot}64k$	$+3{\cdot}78h + 10{\cdot}07k$	$+0{\cdot}07h + 8{\cdot}43k$	6·79
„ „	59·57	$-4{\cdot}89h + 8{\cdot}24k$	$-8{\cdot}40h + 21{\cdot}50k$	$-3{\cdot}51h + 13{\cdot}26k$	7·79
„ „	59·64	$+2{\cdot}78h - 0{\cdot}29k$	$+0{\cdot}22h + 9{\cdot}73k$	$-2{\cdot}56h + 10{\cdot}02k$	5·97
„ 27	59·26	$+11{\cdot}13h - 8{\cdot}11k$	$+2{\cdot}60h + 9{\cdot}01k$	$-8{\cdot}53h + 17{\cdot}12k$	6·88
„ „	59·32	$-1{\cdot}01h + 3{\cdot}16k$	$+1{\cdot}68h + 10{\cdot}20k$	$+2{\cdot}69h + 7{\cdot}04k$	7·75
„ „	59·31	$+14{\cdot}47h - 12{\cdot}33k$	$+4{\cdot}09h + 8{\cdot}46k$	$-10{\cdot}38h + 20{\cdot}79k$	8·34
1864.					
Jan. 12	36·99	$+10{\cdot}95h + 20{\cdot}49k$	$+23{\cdot}86h + 31{\cdot}29k$	$+12{\cdot}91h + 10{\cdot}80k$	18·88
„ „	37·57	$+19{\cdot}22h + 11{\cdot}00k$	$+31{\cdot}18h + 20{\cdot}89k$	$+11{\cdot}96h + 9{\cdot}89k$	17·40
„ „	37·79	$+23{\cdot}38h + 8{\cdot}83k$	$+33{\cdot}29h + 17{\cdot}87k$	$+9{\cdot}91h + 9{\cdot}04k$	15·09
„ 13	37·98	$+20{\cdot}51h + 8{\cdot}38k$	$+33{\cdot}24h + 15{\cdot}68k$	$+12{\cdot}73h + 7{\cdot}30k$	15·95
„ „	38·31	$+15{\cdot}30h + 10{\cdot}96k$	$+28{\cdot}78h + 17{\cdot}31k$	$+13{\cdot}48h + 6{\cdot}35k$	15·79
„ „	38·67	$+16{\cdot}54h + 8{\cdot}31k$	$+23{\cdot}72h + 21{\cdot}81k$	$+7{\cdot}18h + 13{\cdot}50k$	16·48
„ 14	38·92	$+10{\cdot}32h + 13{\cdot}06k$	$+27{\cdot}68h + 16{\cdot}29k$	$+17{\cdot}36h + 3{\cdot}23k$	16·38
„ „	39·17	$+12{\cdot}92h + 9{\cdot}81k$	$+21{\cdot}88h + 20{\cdot}54k$	$+8{\cdot}96h + 10{\cdot}73k$	15·68
„ „	39·39	$+6{\cdot}63h + 15{\cdot}62k$	$+12{\cdot}12h + 30{\cdot}07k$	$+5{\cdot}49h + 14{\cdot}45k$	15·89
„ 15	39·52	$+13{\cdot}62h + 6{\cdot}67k$	$+10{\cdot}58h + 29{\cdot}47k$	$+3{\cdot}04h + 22{\cdot}80k$	15·78

The temperatures in these tables are corrected for errors of thermometers.

COMPARISON OF YARD [$\beta\cdot\gamma$] OF ORDNANCE TOISE WITH THE STANDARD YARD.

Date.	Temp.	OT	Y_{55}	Difference of Length in Micrometer Divisions.	$[\beta\cdot\gamma]-Y_{55}$
1863.					
Oct. 10	61°·85	+ 7·87 h + 11·49 k	+ 11·26 h + 11·76 k	+ 3·39 h + 0·27 k	+ 2·90
„ 12	59·82	+ 50·44 h − 18·61 k	+ 87·03 h − 49·96 k	+ 36·59 h − 31·35 k	4·07
„ „	60·01	+ 43·53 h − 12·54 k	+ 81·86 h − 46·73 k	+ 38·33 h − 34·19 k	3·19
„ „	60·04	+ 99·62 h − 69·03 k	+ 138·51 h − 104·34 k	+ 38·89 h − 35·31 k	2·74
„ 13	59·38	+ 117·66 h − 83·31 k	+ 141·29 k − 102·77 k	+ 23·63 h − 19·46 k	3·26
„ „	59·54	+ 108·09 h − 74·37 h	+ 131·70 h − 94·40 k	+ 23·61 h − 20·03 k	2·79
„ „	59·68	+ 107·34 h − 75·79 h	+ 137·83 h − 100·76 k	+ 30·49 h − 24·97 k	4·31
„ 14	59·85	+ 120·97 h − 90·62 k	+ 158·77 h − 124·10 k	+ 37·80 h − 33·48 k	3·33
„ „	60·07	+ 98·30 h − 69·66 k	+ 149·87 h − 115·37 k	+ 51·57 h − 45·71 k	4·52
„ „	60·20	+ 29·11 h − 0·49 k	+ 60·74 h − 27·96 k	+ 31·63 h − 27·47 k	3·22
„ 28	58·61	− 13·54 h + 17·53 k	+ 10·17 h − 2·09 k	+ 23·71 h − 19·62 k	3·19
„ „	58·60	− 7·42 h + 11·21 k	+ 10·34 h − 2·34 k	+ 17·76 h − 13·55 k	3·31
„ „	58·61	+ 22·31 h − 18·09 k	+ 42·70 h − 34·36 k	+ 20·39 h − 16·27 k	3·23
„ 29	57·94	+ 15·66 h − 7·39 k	+ 18·51 h − 6·56 k	+ 2·85 h + 0·83 k	2·92
„ „	57·96	+ 6·38 h + 0·96 k	+ 36·64 h − 23·41 k	+ 30·26 h − 24·37 k	4·61
„ „	57·96	+ 10·32 h − 2·17 k	+ 35·78 h − 23·22 k	+ 25·46 h − 21·05 k	3·44
„ 30	57·34	+ 15·30 h − 3·57 k	+ 30·40 h − 14·52 k	+ 15·10 h − 10·95 k	3·26
„ „	57·36	+ 2·53 h + 7·48 k	+ 11·67 h + 4·31 k	+ 9·14 h − 3·17 k	4·73
„ „	57·40	+ 7·56 h + 3·58 k	+ 33·33 h − 16·27 k	+ 25·77 h − 19·85 k	4·65
„ 31	56·76	− 1·24 h + 14·82 k	+ 21·26 h − 1·54 k	+ 22·50 h − 16·36 k	4·83
1864.					
Jan. 15	40·18	+ 6·34 h + 16·79 k	+ 15·26 h + 22·59 k	+ 8·92 h + 5·80 k	11·72
„ „	40·26	+ 14·28 h + 7·91 k	+ 23·23 h + 13·98 k	+ 8·95 h + 6·07 k	11·95
„ 16	40·15	+ 6·82 h + 16·57 k	+ 14·57 h + 24·11 k	+ 7·75 h + 7·54 k	12·18
„ „	40·36	+ 9·23 h + 12·15 k	+ 16·20 h + 21·24 k	+ 6·97 h + 9·09 k	12·79
„ „	40·50	+ 17·66 h + 3·02 k	+ 26·24 h + 10·29 k	+ 8·58 h + 7·27 k	12·62
„ 18	40·80	+ 15·03 h + 4·87 k	+ 20·96 h + 14·04 k	+ 5·93 h + 9·17 k	12·03
„ „	41·01	+ 12·07 h + 6·46 k	+ 12·10 h + 21·59 k	+ 0·03 h + 15·13 k	12·09
„ „	41·23	+ 11·61 h + 5·93 k	+ 15·97 h + 16·46 k	+ 4·36 h + 10·53 k	11·86
„ 19	41·34	+ 7·37 h + 10·28 k	+ 15·59 h + 17·26 k	+ 8·22 h + 6·98 k	12·10
„ „	41·59	+ 7·14 h + 8·53 k	+ 10·94 h + 20·49 k	+ 3·80 h + 11·96 k	12·56

Let $x'\,x_{,}$ be the excesses of lengths of the yards [$\alpha\cdot\beta$], [$\beta\cdot\gamma$] above Y_{55} at 62°, $y'\,y_{,}$ the differences of expansion, so that

$$x'+fy',\ x_{,}+fy_{,}$$

are the excesses of lengths of the yards above Y_{55} at the temperature of 62° + f Fahrenheit. The quantity f being given at each comparison, we have thirty equations from which to deduce x' and y', and as many for $x_,$ and $y_,$. These last equations, viz., in $x_,\ y_,$ being treated according to the method of least squares, resolve themselves into the following:

$$\begin{aligned} 30\,x_, - 273{\cdot}60\,y_, - 194{\cdot}40 &= 0 \\ -273{\cdot}60\,x_, + 4739{\cdot}51\,y_, + 2820{\cdot}58 &= 0 \end{aligned} \tag{1}$$

which give

$$\begin{aligned} x_, &= +\,2{\cdot}223 \\ y_, &= -\,0{\cdot}4668 \end{aligned} \tag{2}$$

The comparisons of the right yard afford the following:

$$\begin{aligned} 30\,x' - 269{\cdot}45\,y' - 304{\cdot}57 &= 0 \\ -269{\cdot}45\,x' + 5637{\cdot}54\,y' + 4100{\cdot}94 &= 0 \end{aligned} \tag{3}$$

where x' is the excess of the length of the right yard above Y_{65} at 62°, and y' the rate of relative expansion. The values resulting from these equations are

$$\begin{aligned} x' &= +\,6{\cdot}341 \\ y' &= -\,0{\cdot}4244 \end{aligned} \tag{4}$$

From this it would appear as though one half of the bar **OT** had a slightly different rate of expansion from the other half. But this may well be attributed to errors in the operation, as the difference is small. If we assume, as is indeed necessary, that the rate of expansion is the same for the right yard as for the left, then the sixty comparisons must be combined in one series of equations containing three unknown quantities $x_,\ x'$ and y.

The resulting equations are as follows:—

$$\begin{aligned} 30\,x_, - 273{\cdot}60\,y - 194{\cdot}40 &= 0 \\ 30\,x' - 269{\cdot}45\,y - 304{\cdot}57 &= 0 \\ -273{\cdot}60\,x_, - 269{\cdot}45\,x' + 10377{\cdot}05\,y + 6921{\cdot}52 &= 0 \end{aligned} \tag{5}$$

If we write A B C for the absolute terms of these equations we have by elimination

$$\begin{aligned} x_, + 0{\cdot}0485620\,A + 0{\cdot}0149977\,B + 0{\cdot}0016698\,C &= 0 \\ x' + 0{\cdot}0149977\,A + 0{\cdot}0481035\,B + 0{\cdot}0016445\,C &= 0 \\ y + 0{\cdot}0016698\,A + 0{\cdot}0016445\,B + 0{\cdot}0001831\,C &= 0 \end{aligned} \tag{6}$$

While the numerical values are found to be

$$\begin{aligned} x_, &= +\,2{\cdot}451 \\ x' &= +\,6{\cdot}184 \\ y &= -\,0{\cdot}4418 \end{aligned} \tag{7}$$

If from equations (6) we write out the expression for the reciprocal of the weight of $\alpha x_, + \beta x' + \gamma y$ and then make $\alpha = 1$, $\beta = 1$, we get the reciprocal of the weight of $x_, + x' + \gamma y$

$$\cdot 12666 + \cdot 00663\,\gamma + \cdot 000183\,\gamma^2 \tag{8}$$

Now the length of the two yards taken together is at the temperature $62° + f$ Fahrenheit

$$x_, + x' + 2\,fy$$

in excess of $2\,Y_{55}$ at the same temperature; and the weight of the determination at that temperature is the reciprocal of

$$\cdot 12666 + \cdot 01326 f + \cdot 000732 f^2 \qquad (9)$$

The following Table contains in contrast the observed differences of length, and the differences computed from the formulæ $x_{,} + (t - 62)\,y$, $x' + (t - 62)\,y$.

COMPARISON OF YARDS OF ORDNANCE TOISE WITH THE STANDARD YARD No. 55.

$[\beta\cdot\gamma] - Y_{55}$			$[\alpha\cdot\beta] - Y_{55}$		
Observed.	Computed.	Error.	Observed.	Computed.	Error.
2·90	2·52	+ 0·38	6·66	6·63	+ 0·03
4·07	3·41	+ 0·66	6·41	7·18	− 0·77
3·19	3·33	− 0·14	8·32	7·20	+ 1·12
2·74	3·32	− 0·58	7·68	7·26	+ 0·42
3·26	3·61	− 0·35	8·63	7·66	+ 0·97
2·79	3·54	− 0·75	5·60	6·18	− 0·58
4·31	3·47	+ 0·84	5·67	6·69	− 1·02
3·33	3·40	− 0·07	7·13	6·68	+ 0·45
4·52	3·30	+ 1·22	5·85	6·68	− 0·83
3·22	3·25	− 0·03	6·70	6·68	+ 0·02
3·19	3·95	− 0·76	7·22	6·13	+ 1·09
3·31	3·95	− 0·64	7·92	6·51	+ 1·41
3·23	3·95	− 0·72	6·24	6·53	− 0·29
2·92	4·24	− 1·32	7·70	6·56	+ 1·14
4·61	4·23	+ 0·38	6·79	7·35	− 0·56
3·44	4·23	− 0·79	7·79	7·25	+ 0·54
3·26	4·51	− 1·25	5·97	7·22	− 1·25
4·73	4·50	+ 0·23	6·88	7·39	− 0·51
4·65	4·48	+ 0·17	7·75	7·36	+ 0·39
4·83	4·77	+ 0·06	8·34	7·37	+ 0·97
11·72	12·09	− 0·37	18·88	17·23	+ 1·65
11·95	12·05	− 0·10	17·40	16·97	+ 0·43
12·18	12·10	+ 0·08	15·09	16·88	− 1·79
12·79	12·01	+ 0·78	15·95	16·79	− 0·84
12·62	11·95	+ 0·67	15·79	16·65	− 0·86
12·03	11·82	+ 0·21	16·48	16·49	− 0·01
12·09	11·72	+ 0·37	16·38	16·38	+ 0·00
11·86	11·63	+ 0·23	15·68	16·27	− 0·59
12·10	11·58	+ 0·52	15·89	16·17	− 0·28
12·56	11·47	+ 1·09	15·78	16·11	− 0·33

The sum of the squares of the differences or errors is 33·75, consequently the probable error of a single comparison is

$$\pm \cdot 674 \sqrt{\frac{33\cdot 75}{60-3}} = \pm \cdot 518 \qquad (10)$$

The probable error of $x_{,} + x'$ is consequently

$$\pm \cdot 518 \sqrt{\cdot 12666} = \pm\, 0\cdot 184 \qquad (11)$$

The probable error of y is

$$\pm \cdot 518 \sqrt{\cdot 000183} = \pm 0{\cdot}0070 \qquad (12)$$

Finally, the lengths of the two yards of **OT** in terms of $\mathbf{Y}_{55}$ at the temperature t are

$$\left.\begin{aligned} [\beta \cdot \gamma] &= \mathbf{Y}_{55} + 2 \cdot 45 - 0 \cdot 4418\,(t - 62) \\ [\alpha \cdot \beta] &= \mathbf{Y}_{55} + 6 \cdot 18 - 0 \cdot 4418\,(t - 62) \end{aligned}\right\} \qquad (13)$$

2.

The small space of 4·74 inches on the right of the Ordnance Toise was compared on three successive days with the space $[\tau \cdot f]$ of **OF**, whose value we have already determined in terms of standard yard No. 55. The individual comparisons, three in each visit, recorded in exactly the same form as at page 97, are shown in the following Table:

COMPARISON OF SPACE $[\gamma \cdot \delta]$ ON ORDNANCE TOISE WITH $[\tau \cdot f]$

Date.	Temp.	$[\gamma \cdot \delta]$	$[f \cdot \tau]$	Difference of Length in Micrometer Divisions.	$[\gamma \cdot \delta] - [f \cdot \tau]$
1863.	°				
Oct. 15	62·48	+ 28·40 h − 25·83 k	− 21·33 h + 28·53 k	− 49·73 h + 54·36 k	+3·85
” ”	62·48	− 8·43 h + 11·10 k	− 20·87 h + 28·23 k	− 12·44 h + 17·13 k	3·78
” ”	62·48	− 7·77 h + 11·50 k	− 25·87 h + 32·37 k	− 18·10 h + 20·87 k	2·26
” ”	62·76	+ 4·03 h − 2·50 k	− 25·40 h + 31·13 k	− 29·43 h + 33·63 k	3·45
” ”	62·76	+ 4·43 h − 1·90 k	− 3·53 h + 10·27 k	− 7·96 h + 12·17 k	3·39
” ”	62·76	− 4·53 h + 6·90 k	− 4·00 h + 10·47 k	+ 0·53 h + 3·57 k	3·27
” 16	62·26	+ 4·23 h − 0·07	− 35·23 h + 42·37 k	− 39·46 h + 42·44 k	2·50
” ”	62·26	− 50·50 h + 53·47 k	− 35·13 h + 42·73 k	+ 15·37 h − 10·74 k	3·65
” ”	62·26	− 50·37 h + 53·40 k	− 96·27 h + 103·70 k	− 45·90 h + 50·30 k	3·65
” ”	62·33	− 127·03 h + 129·00 k	− 97·00 h + 103·27 k	+ 30·03 h − 25·73 k	3·34
” ”	62·33	− 128·13 h + 128·90 k	− 87·33 h + 92·50 k	+ 40·80 h − 36·40 k	3·38
” ”	62·33	− 116·87 h + 118·17 k	− 86·07 h + 92·77 k	+ 30·80 h − 25·40 k	4·21
” ”	62·41	− 117·50 h + 120·43 k	− 66·23 h + 72·53 k	+ 51·27 h − 47·90 k	2·53
” ”	62·41	− 111·93 h + 114·67 k	− 65·60 h + 72·67 k	+ 46·33 h − 42·00 k	3·31
” ”	62·41	− 111·87 h + 114·80 k	− 62·83 h + 69·50 k	+ 49·04 h − 45·30 k	2·83
” ”	62·53	− 112·00 h + 114·27 k	− 72·63 h + 80·23 k	+ 39·37 h − 34·04 k	4·14
” ”	62·53	− 112·10 h + 114·10 k	− 80·50 h + 87·73 k	+ 31·60 h − 26·37 k	4·08
” ”	62·53	− 101·67 h + 105·17 k	− 80·10 h + 87·67 k	+ 21·57 h − 17·50 k	3·19
” ”	62·71	− 105·73 h + 107·47 k	− 64·40 h + 70·97 k	+ 41·33 h − 36·50 k	3·72
” ”	62·71	− 79·93 h + 82·63 k	− 64·20 h + 71·20 k	+ 15·73 h − 11·43 k	3·38
” ”	62·71	− 80·30 h + 82·40 k	− 76·80 h + 83·20 k	+ 3·50 h + 0·80 k	3·41
” 17	62·54	− 158·70 h + 159·67 k	− 50·57 h + 57·07 k	+ 108·13 h − 102·60 k	4·09
” ”	62·54	− 158·20 h + 159·70 k	− 98·70 h + 103·93 k	+ 59·50 h − 55·77 k	2·80
” ”	62·54	− 168·90 h + 170·13 k	− 98·10 h + 103·83 k	+ 70·80 h − 66·30 k	3·37
” ”	62·59	− 173·47 h + 173·43 k	− 115·03 h + 118·67 k	+ 58·44 h − 54·76 k	2·76
” ”	62·59	− 149·03 h + 149·33 k	− 114·70 h + 119·40 k	+ 34·33 h − 29·93 k	3·41
” ”	62·59	− 149·53 h + 149·93 k	− 109·57 h + 114·50 k	+ 39·96 h − 35·43 k	3·50
” ”	62·72	− 165·37 h + 167·07 k	− 109·50 h + 114·40 k	+ 55·87 h − 52·67 k	2·38
” ”	62·72	− 165·93 h + 166·40 k	− 102·60 h + 107·50 k	+ 63·33 h − 58·90 k	3·34
” ”	62·72	− 159·33 h + 159·33 k	− 102·90 h + 107·00 k	+ 56·43 h − 52·33 k	3·10

From this we find for the space on **OT** the value

$$[\gamma \cdot \delta] = [\tau \cdot f] + 3 \cdot 34$$

at the temperature 62°·53. The sum of the squares of the errors is 7·875, consequently the probable error of a single comparison is

$$\pm \cdot 674 \sqrt{\frac{7 \cdot 875}{30 - 1}} = \pm 0 \cdot 351 \tag{14}$$

and the probable error of the mean of 30 determinations

$$\pm \frac{\cdot 351}{\sqrt{30}} = \pm 0 \cdot 064 \tag{15}$$

But we require the difference of length at 62°. We have seen by equations (7) and (8), page 77, that

$$\mathbf{F} = \tfrac{1}{3} \mathbf{Y}_{55} - 0 \cdot 36 + 0 \cdot 0066\,(t - 62)$$

with the probable error

$$\pm \left\{ \cdot 011715 + \cdot 001256\,(t - 62) + \cdot 0000488\,(t - 62)^2 \right\}^{\frac{1}{2}}$$

Therefore

$$\frac{474}{1200}\,\mathbf{F} = \frac{474}{3600}\,\mathbf{Y}_{55} - 0 \cdot 14 + \cdot 0026\,(t - 62)$$

with a probable error

$$\pm \left\{ \cdot 001828 + \cdot 000196\,(t - 62) + \cdot 0000076\,(t - 62)^2 \right\}^{\frac{1}{2}}$$

Also by equation 46, page 71, we have

$$[\tau \cdot f] = \frac{474}{1200}\,\mathbf{F} - 3 \cdot 10 \pm 0 \cdot 119$$

Hence

$$[\tau \cdot f] = \frac{474}{3600}\,\mathbf{Y}_{55} - 3 \cdot 24 + \cdot 0026\,(t - 62)$$

with a probable error at the temperature t of

$$\pm \left\{ \cdot 015989 + \cdot 000196\,(t - 62) + \cdot 0000076\,(t - 62)^2 \right\}^{\frac{1}{2}}$$

Now let the small space on **OT** at 62° be equal to $\frac{474}{3600}\,\mathbf{Y}_{55} + u$, then at the temperature t its length will be

$$[\gamma \cdot \delta] = \frac{474}{3600}\,\mathbf{Y}_{55} + u + \frac{474}{3600}\,y\,(t - 62)$$

where y is the excess of the expansion of a yard of **OT** above the expansion of $\mathbf{Y}_{55}$ for 1° Fahrenheit.

Now this space exceeds $[\tau \cdot f]$ by

$$u + 3 \cdot 24 + \left[\frac{474}{3600}\,y - \cdot 0026\right](t - 62)$$

and the observed value (for $t = 62^\circ \cdot 53$) we have just found to be $3 \cdot 34 \pm 0 \cdot 064$; consequently

$$u = 3 \cdot 34 - 3 \cdot 24 + \cdot 0026 \, (t - 62) - \frac{474}{3600} y \, (t - 62)$$

Here $y = - \cdot 4418 \pm \cdot 0070$ and

$$\frac{474}{3600} y = - \cdot 0582 \pm \cdot 0009$$

Consequently we have

$$x = + 0 \cdot 13$$

The probable error of u is composed of three independent parts; namely, first, that of the quantity $3 \cdot 34$; second, that of $[\tau \cdot f]$ or $- 3 \cdot 24 + \cdot 0026 \, (t - 62)$, when $t = 62 \cdot 53$; and, third, that of the fraction of y. This last, however, becomes insensible. Thus the total probable error of u

$$\pm \left\{ \cdot 004096 + \cdot 015989 + \cdot 000103 + \cdot 000002 \right\}^{\frac{1}{2}} = \pm \, 0 \cdot 142$$

3.

The three spaces of which the Ordnance Toise is composed are then as follows:

$$[\alpha \cdot \beta] = \quad Y_{55} + 6 \cdot 18$$
$$[\beta \cdot \gamma] = \quad Y_{55} + 2 \cdot 45$$
$$[\gamma \cdot \delta] = \frac{474}{3600} Y_{55} + 0 \cdot 13$$

the sum of which is

$$[\alpha \cdot \delta] = \frac{7674}{3600} Y_{55} + 8 \cdot 76 \pm \sqrt{(\cdot 142)^2 + (\cdot 184)^2}$$

As in the case of the bar **OF** we have used the letter **F** as an algebraic symbol for the length of the bar, so now we shall use T_0 to represent the length of the Ordnance Toise. We have then,

$$T_0 = (2 \cdot 13167543 \pm \cdot 00000023) \, Y_{55}$$

both bars* being at the temperature of 62° Fahrenheit.

The total number of micrometer readings from which this result is obtained is 2520.

* Further observations on the length of T_0 will be found at page 139.

IX.

DETERMINATION OF THE LENGTH OF THE ORDNANCE METRE.

This bar, which is exactly the same in section as the Ordnance Toise, carries on its upper surface three disks *a*, *b*, *c* of platinum, so defining by lines the length of the *yard* and *metre*, or at least very approximately those lengths. The determination of the length of the bar therefore divides itself into two parts; first, the determination of the true length of the yard space $[a \cdot b]$; and, secondly, the determination of the length of the small space $[b \cdot c]$ of 3·38 inches.

The yard on the Ordnance Metre, which is designated **OM**, was compared with Y_{55} on the following days, March 8, 9, 10, 11, 12, 14; July 5, 6, 7, 25, 26, 27, 28; in all forty comparisons or visits. Of these, fifteen were at a temperature averaging about 46°, and the remainder at an average temperature of about 64°. Thus both the length at 62° and the relative expansion are well determined. In these comparisons the yard Y_{55} lay in the same box with **OM**, the bars occupying alternately the inner and outer position. The focal adjustment was renewed at nearly every comparison. The following Table contains the result of each visit. The temperatures are corrected for errors of thermometers.

COMPARISONS OF ORDNANCE METRE AND STANDARD YARD NO. 55.

Date.	Temp.	Y_{55}	OM	Difference in Micrometer Divisions.	$[a \cdot b] - Y_{55}$
1864.	°				
March 8	47·91	21·00 *h* + 21·48 *k*	23·75 *h* + 12·68 *k*	− 2·75 *h* + 8·80 *k*	+ 4·84
„ 9	47·59	26·92 *h* + 18·48 *k*	19·66 *h* + 20·00 *k*	+ 7·26 *h* − 1·52 *k*	+ 4·56
„ „	47·65	24·68 *h* + 20·12 *k*	17·80 *h* + 20·48 *k*	+ 6·88 *h* − 0·36 *k*	+ 5·18
„ „	47·55	22·40 *h* + 22·78 *k*	25·00 *h* + 14·30 *k*	− 2·60 *h* + 8·48 *k*	+ 4·70
„ 10	46·85	27·68 *h* + 21·74 *k*	23·18 *h* + 20·02 *k*	+ 4·50 *h* + 1·72 *k*	+ 4·95
„ „	46·78	30·20 *h* + 20·15 *k*	27·78 *h* + 16·15 *k*	+ 2·42 *h* + 4·00 *k*	+ 5·12
„ „	46·62	28·58 *h* + 21·53 *k*	18·15 *h* + 25·52 *k*	+ 10·43 *h* − 3·99 *k*	+ 5·11
„ 11	46·00	31·90 *h* + 23·35 *k*	24·52 *h* + 23·72 *k*	+ 7·38 *h* − 0·37 *k*	+ 5·57
„ „	46·02	32·85 *h* + 21·36 *k*	32·05 *h* + 14·98 *k*	+ 0·80 *h* + 6·38 *k*	+ 5·73
„ „	45·98	27·82 *h* + 25·92 *k*	21·10 *h* + 25·75 *k*	+ 6·72 *h* + 0·17 *k*	+ 5·48
„ 12	45·71	25·72 *h* + 30·42 *k*	18·48 *h* + 30·05 *k*	+ 7·24 *h* + 0·37 *k*	+ 6·05
„ „	45·87	26·05 *h* + 28·93 *k*	24·05 *h* + 24·85 *k*	+ 2·00 *h* + 4·08 *k*	+ 4·85
„ „	45·82	23·78 *h* + 30·98 *k*	29·50 *h* + 18·57 *k*	− 5·72 *h* + 12·41 *k*	+ 5·36
„ 14	45·39	23·43 *h* + 33·80 *k*	22·18 *h* + 27·83 *k*	+ 1·25 *h* + 5·97 *k*	+ 5·76
„ „	45·55	23·48 *h* + 30·93 *k*	21·63 *h* + 26·36 *k*	+ 1·85 *h* + 4·57 *k*	+ 5·12

COMPARISONS OF ORDNANCE METRE AND STANDARD YARD No. 55—*continued.*

Date.	Temp.	Y_{55}	OM	Difference in Micrometer Divisions.	$[a \cdot b] - Y_{55}$
1864.	°				
July 5	61·19	22·50 *h* + 28·05 *k*	28·70 *h* + 24·50 *k*	− 6·20 *h* + 3·55 *k*	− 2·10
„ 6	61·12	24·60 *h* + 26·25 *k*	26·05 *h* + 26·78 *k*	− 1·45 − 0·53 *k*	− 1·58
„ „	61·28	29·93 *h* + 20·38 *k*	30·35 *h* + 21·25 *k*	− 0·42 *h* − 0·87 *k*	− 1·03
„ „	61·38	29·90 *h* + 19·60 *k*	29·28 *h* + 21·58 *k*	+ 0·62 *h* − 1·98 *k*	− 1·09
„ „	61·50	24·10 *h* + 24·93 *k*	28·33 *h* + 22·30 *k*	− 4·23 *h* + 2·63 *k*	− 1·26
„ 7	61·48	24·73 *h* + 24·25 *k*	24·18 *h* + 26·48 *k*	+ 0·55 *h* − 2·23 *k*	− 1·34
„ „	61·65	26·55 *h* + 21·65 *k*	27·20 *h* + 22·33 *k*	− 0·65 *h* − 0·68 *k*	− 1·06
„ „	61·75	25·65 *h* + 21·13 *k*	22·78 *h* + 25·75 *k*	+ 2·87 *h* − 4·62 *k*	− 1·41
„ „	61·85	23·88 *h* + 22·28 *k*	24·88 *h* + 23·35 *k*	− 1·00 *h* − 1·07 *k*	− 1·65
„ 25	65·38	24·93 *h* + 29·50 *k*	32·85 *h* + 25·10 *k*	− 7·92 *h* + 4·40 *k*	− 2·78
„ „	65·51	24·05 *h* + 28·63 *k*	28·98 *h* + 27·60 *k*	− 4·93 *h* + 1·03 *k*	− 3·10
„ „	65·62	24·48 *h* + 27·63 *k*	27·40 *h* + 29·73 *k*	− 2·92 *h* − 2·10 *k*	− 4·00
„ „	65·62	24·65 *h* + 27·00 *k*	27·30 *h* + 28·00 *k*	− 2·65 *h* − 1·00 *k*	− 2·90
„ 26	65·01	27·10 *h* + 27·90 *k*	28·38 *h* + 28·47 *k*	− 1·28 *h* − 0·57 *k*	− 1·47
„ „	65·14	21·33 *h* + 33·58 *k*	27·33 *h* + 29·67 *k*	− 6·00 *h* + 3·91 *k*	− 1·65
„ „	65·39	29·35 *h* + 23·45 *k*	30·90 *h* + 24·18 *k*	− 1·55 *h* − 0·73 *k*	− 1·81
„ 27	65·23	29·98 *h* + 24·25 *k*	29·88 *h* + 26·12 *k*	+ 0·10 *h* − 1·87 *k*	− 1·41
„ „	65·31	22·65 *h* + 31·28 *k*	24·05 *h* + 32·68 *k*	− 1·40 *h* − 1·40 *k*	− 2·23
„ „	65·37	27·85 *h* + 24·60 *k*	24·55 *h* + 31·15 *k*	+ 3·30 *h* − 6·55 *k*	− 2·60
„ „	65·43	26·40 *h* + 25·23 *k*	28·75 *h* + 28·08 *k*	− 2·35 *h* − 2·85 *k*	− 4·14
„ „	65·43	24·23 *h* + 28·45 *k*	27·15 *h* + 28·70 *k*	− 2·92 *h* − 0·25 *k*	− 2·52
„ 28	65·27	23·70 *h* + 29·53 *k*	24·83 *h* + 30·55	− 1·13 *h* − 1·02 *k*	− 1·71
„ „	65·33	26·97 *h* + 26·63 *h*	28·40 *h* + 27·25 *k*	− 1·43 *h* − 0·62 *k*	− 1·63
„ „	65·38	26·48 *h* + 27·32 *k*	26·68 *h* + 29·37 *k*	− 0·20 *h* − 2·05 *k*	− 1·79
„ „	65·44	24·45 *h* + 27·03 *k*	25·70 *h* + 29·08 *k*	− 1·25 *h* − 2·05 *k*	− 2·63

The irregularity of the results during the last few days is greater than usual; it attracted special attention at the time of making the observations, but no explanation could be perceived. This is the more remarkable as the temperature was unusually steady during the four days in question.

Let the excess of the yard on **OM** above Y_{55} be, at 62°, $= x$, and let the excess of the expansion of a yard of **OM** for 1° above the expansion of Y_{55} be $= y$, then at any temperature t the length of $[a \cdot b] = Y_{55} + x + (t - 62)\, y$. Now we have in the above Table forty observed differences of length at given temperatures, and these give forty equations from which to ascertain x and y.

Solving these equations by the method of least squares, there results;

$$40 \cdot 00\, x - 183 \cdot 650\, y - 27 \cdot 490 = 0$$
$$- 183 \cdot 65\, x + 3805 \cdot 187\, y + 1343 \cdot 395 = 0 \qquad (1)$$

If we put A and B for the absolute terms we get

$$x + \cdot 0321167\, A + \cdot 0015500\, B = 0$$
$$y + \cdot 0015500\, A + \cdot 0003376\, B = 0 \qquad (2)$$

restoring the values of A and B

$$x = - 1 \cdot 199$$
$$y = - 0 \cdot 4109 \qquad (3)$$

The length, therefore, of the yard on **OM** at the temperature t is

$$[a \cdot b] = Y_{55} - 1 \cdot 20 - 0 \cdot 4109\,(t - 62) \tag{4}$$

The reciprocals of the weights of the determinations of x and y are, from equation (2),

$$x \ldots\ldots 0 \cdot 03212$$
$$y \ldots\ldots 0 \cdot 00034 \tag{5}$$

Substituting the values of x and y in the equations of condition, we obtain the errors of the different comparisons, as shown in the following Table.

TABLE OF ERRORS OF COMPARISONS OF THE YARD ON **OM** WITH Y_{55}.

Date.	Error.	Date.	Error.	Date.	Error.	Date.	Error.
March 8	+ 0·25	March 12	+ 0·56	July 7	− 0·35	July 26	+ 0·78
„ 9	− 0·16	„ „	− 0·58	„ „	+ 0·00	„ 27	+ 1·12
„ „	+ 0·48	„ „	− 0·09	„ „	− 0·31	„ „	+ 0·33
„ „	− 0·04	„ 14	+ 0·13	„ „	− 0·51	„ „	− 0·01
„ 10	− 0·08	„ „	− 0·44	„ 25	− 0·19	„ „	− 1·53
„ „	+ 0·07	July 5	− 1·23	„ „	− 0·46	„ „	+ 0·09
„ „	− 0·01	„ 6	− 0·74	„ „	− 1·31	„ 28	+ 0·83
„ 11	+ 0·20	„ „	− 0·13	„ „	− 0·21	„ „	+ 0·94
„ „	+ 0·36	„ „	− 0·14	„ 26	+ 0·97	„ „	+ 0·80
„ „	+ 0·10	„ „	− 0·27	„ „	+ 0·84	„ „	− 0·02

$\Sigma\,(e^2) = 14 \cdot 2164.$

Hence we find the probable error of a single comparison to be

$$\pm \cdot 674 \sqrt{\frac{14 \cdot 216}{40 - 2}} = \pm 0 \cdot 412 \tag{6}$$

The probable error, therefore, of x is

$$\pm 0 \cdot 412 \sqrt{\cdot 03212} = \pm 0 \cdot 074 \tag{7}$$

And the probable error of y

$$\pm 0 \cdot 412 \sqrt{\cdot 00034} = \pm 0 \cdot 0076 \tag{8}$$

From equations (2), if required, is easily obtained the weight of the determination of the length of $[a \cdot b]$ at any given temperature.

2

The small space $[b \cdot c]$ of $3 \cdot 38$ inches was compared with the corresponding space on **OF**, $[u \cdot e]$, on August 4th, 5th, and 6th.

The individual or single comparisons, generally three in number at each visit, and recorded as at page 97, are shown in the following Table.

COMPARISONS OF 3·38 INCH SPACE ON ORDNANCE METRE AND ORDNANCE FOOT.

Date.		Temp.	$[b\cdot c]$	$[\mu\cdot e]$	Difference in Micrometer Divisions.	$[\mu\cdot e]-[b\cdot c]$
1864.		°				
Aug.	4	65·40	96·33 *h* + 89·50 *k*	8·77 *h* + 10·80 *k*	87·56 *h* + 78·70 *k*	132·40
"	"	65·41	92·13 *h* + 93·43 *k*	8·77 *h* + 10·40 *k*	83·36 *h* + 83·03 *k*	132·52
"	5	64·70	92·33 *h* + 93·53 *k*	9·37 *h* + 9·33 *k*	82·96 *h* + 84·20 *k*	133·14
"	"	64·70	96·20 *h* + 89·73 *k*	10·03 *h* + 9·30 *k*	86·17 *h* + 80·43 *k*	132·68
"	"	64·70	91·97 *h* + 93·17 *k*	11·47 *h* + 7·50 *k*	80·50 *h* + 85·67 *k*	132·35
"	"	65·00	90·57 *h* + 94·43 *k*	9·23 *h* + 8·73 *k*	81·34 *h* + 85·70 *k*	133·05
"	"	65·00	88·53 *h* + 97·20 *k*	9·20 *h* + 8·90 *k*	79·33 *h* + 88·30 *k*	133·52
"	"	65·00	90·63 *h* + 95·47 *k*	11·73 *h* + 6·50 *k*	78·90 *h* + 88·97 *k*	133·72
"	"	65·20	93·40 *h* + 92·77 *k*	9·93 *h* + 9·17 *k*	83·47 *h* + 83·60 *k*	133·06
"	"	65·20	92·93 *h* + 92·87 *k*	10·63 *h* + 7·30 *k*	82·30 *h* + 85·57 *k*	133·71
"	"	65·20	94·00 *h* + 90·70 *k*	8·37 *h* + 10·63 *k*	85·63 *h* + 80·07 *k*	131·96
"	"	65·45	95·33 *h* + 89·53 *k*	10·33 *h* + 7·97 *k*	85·00 *h* + 81·56 *k*	132·65
"	"	65·45	94·03 *h* + 91·80 *k*	10·67 *h* + 7·53 *k*	83·36 *h* + 84·27 *k*	133·51
"	"	65·45	94·47 *h* + 91·40 *k*	11·07 *h* + 7·87 *k*	83·40 *h* + 83·53 *k*	132·95
"	6	65·66	93·03 *h* + 93·10 *k*	10·77 *h* + 7·40 *k*	82·26 *h* + 85·70 *k*	133·78
"	"	65·66	90·73 *h* + 93·77 *k*	9·90 *h* + 8·57 *k*	80·83 *h* + 85·20 *k*	132·24
"	"	65·66	91·67 *h* + 92·97 *k*	10·03 *h* + 8·33 *k*	81·64 *h* + 84·64 *k*	132·44
"	"	65·85	93·13 *h* + 92·87 *k*	8·00 *h* + 9·93 *k*	85·13 *h* + 82·94 *k*	133·86
"	"	65·85	91·77 *h* + 93·80 *k*	7·53 *h* + 10·23 *k*	84·24 *h* + 83·57 *k*	133·65
"	"	65·85	92·10 *h* + 93·17 *k*	9·63 *h* + 7·80 *k*	82·47 *h* + 85·37 *k*	133·68

From this we find for $[b\cdot c]$ the value

$$[\mu\, e] - 133\cdot 04 \tag{9}$$

at the mean temperature 65°·32. The sum of the squares of the errors, or differences of the individual results from the mean, is 7·0011, consequently the probable error of a single comparison is

$$\pm\cdot 674\sqrt{\frac{7\cdot 00}{20-1}} = \pm\, 0\cdot 409 \tag{10}$$

and the probable error of the mean of 20 comparisons

$$\pm\,\frac{0\cdot 409}{\sqrt{20}} = \pm\, 0\cdot 091 \tag{11}$$

We have next to reduce this to the temperature of 62°. By equations (7) and (8), page 77, it appears that

$$\mathsf{F} = \tfrac{1}{3}\,\mathsf{Y}_{55} - 0\cdot 36 + 0\cdot 0066\,(t-62)$$

with the probable error

$$\pm\left\{\cdot 011715 + \cdot 001256\,(t-62) + \cdot 0000488\,(t-62)^2\right\}^{\frac{1}{2}} \tag{12}$$

consequently

$$\frac{338}{1200}\,\mathsf{F} = \frac{338}{3600}\,\mathsf{Y}_{55} - 0\cdot 10 + \cdot 0019\,(t-62)$$

with the probable error of

$$\pm\{\cdot000929 + \cdot000100\,(t-62) + \cdot0000039\,(t-62)^2\}^{\frac{1}{2}} \tag{13}$$

Also by equation (47), page 71, we have

$$[\mu\cdot e] = \frac{338}{1200}\,\mathbf{F} - 1\cdot24 \pm 0\cdot098$$

Hence

$$[\mu\cdot e] = \frac{338}{3600}\,\mathbf{Y}_{65} - 1\cdot34 + 0\cdot0019\,(t-62) \tag{14}$$

with a probable error of

$$\pm\{\cdot010533 + \cdot000100\,(t-62) + \cdot0000039\,(t-62)^2\}^{\frac{1}{2}} \tag{15}$$

Let now the small space on **OM** at 62° be

$$[b\cdot c] = \frac{338}{3600}\,\mathbf{Y}_{65} + u \tag{16}$$

then at the temperature t its length will be

$$[b\cdot c] = \frac{338}{3600}\,\mathbf{Y}_{65} + u + \frac{338}{3600}\,y\,(t-62) \tag{17}$$

where y is the excess of the expansion, for 1° Fahrenheit, of one yard of **OM** above that of $\mathbf{Y}_{65}$

Now taking the difference of equations (14) and (17), this space exceeds $[\mu\cdot e]$ by

$$u + 1\cdot34 - 0\cdot0019\,(t-62) + \frac{338}{3600}\,y\,(t-62) \tag{18}$$

and the observed value of this quantity for $t = 65^\circ\cdot32$ we have just found to be

$$-133\cdot04 \pm 0\cdot091$$

Consequently

$$u = -133\cdot04 - 1\cdot34 + \cdot0019\,(t-62) - \frac{338}{3600}\,y\,(t-62) \tag{19}$$

Here we must put $t - 62 = 3\cdot32$ and $y = -0\cdot4109$ which gives

$$u = -134\cdot25 \tag{20}$$

The probable error of the expression for u we see to be made up of three independent components: first, the probable error of the quantity 133·04; second, the probable error of the quantity $-1\cdot34 + \cdot0019\,(t-62)$; third, the probable error of a fraction of y, namely,

$$\frac{338}{3600} \times 3\cdot32\,y = 0\cdot3117\,y$$

Now we have the probable error of y by equation (8) equal to $\pm\,0\cdot0076$. So that the probable error of the above fraction of y becomes insignificant.

We have therefore for u the probable error

$$\pm\{\cdot008281 + \cdot010533 + \cdot000332 + \cdot000043\}^{\frac{1}{2}}$$
$$= \pm\,0\cdot139$$

3.

The length of **OM** is composed of the two parts whose lengths we have just determined, namely, the *yard* whose value at 62° is, by (4),

$$\mathbf{Y}_{55} + x = \mathbf{Y}_{55} - 1\cdot 20$$

and the small space whose length at 62° is, by (20),

$$\frac{338}{3600}\mathbf{Y}_{55} + u = \frac{338}{3600}\mathbf{Y} - 134\cdot 25$$

The sum of which is

$$\frac{3938}{3600}\mathbf{Y}_{55} + u + x = \frac{3938}{3600}\mathbf{Y}_{55} - 135\cdot 45 \qquad (21)$$

In order to obtain the probable error of this result it is necessary to remember that u involves y, which is determined from the same observations as x. The quantity $u + x$ is made up as follows:

$$\{-133\cdot 04\} + \{-1\cdot 34 + \cdot 0019 \times 3\cdot 32\} + \{x - \cdot 3117\, y\}$$

The probable error of the directly observed quantity within the first bracket is, by equation (11), $\pm\cdot 091$; that within the second bracket is, by equation (15), $\pm\cdot 104$; that within the third bracket is, by equations (6) and (2),

$$\pm 0\cdot 412\,\{\cdot 032117 + \cdot 003100\,(\cdot 31) + \cdot 000338\,(\cdot 31)^2\}^{\frac{1}{2}} = \pm\cdot 075$$

which, however, does not differ sensibly from the probable error of x alone. Hence for the probable error of $u + x$ we have

$$\pm\{(\cdot 091)^2 + (\cdot 104)^2 + (\cdot 075)^2\}^{\frac{1}{2}} = \pm 0\cdot 158 \qquad (22)$$

We have then finally, from equations (21) (22), for the length of the Ordnance Metre at 62° Fahrenheit

$$\mathbf{OM} = (1\cdot 09375344 \pm \cdot 00000016)\,\mathbf{Y}_{60}$$

X.

ON THE MODE OF SUPPORT OF THE TOISES, Nos. 10, 11.

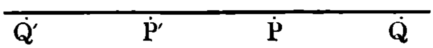
We have described each of these as a flat bar of steel 1·70 inches in breadth, and 0·39 inch thick. The extremities are turned into cylinders having one and the same axis, namely, the line passing through the centres of all transverse sections of the bar. The length of each cylinder is 0·6 of an inch, and its diameter coincides with the thickness of the bar. Omitting from further consideration these cylinders, the length of the bar with section as given above is 75·5 inches.

The bar is supported in its case on four points or rather convex metallic surfaces at equal distances of 21·5 inches apart. These surfaces are of course supposed to have a common tangent plane: if one of the surfaces was above or below the common tangent plane of the other three, the bar would be improperly supported. It appears, as far as can be ascertained, that in all comparisons it has been supported on four rollers having contact with the bar at the same points as when it lies in its case. It is very necessary that these rollers should be in one and the same horizontal plane, and as there may be some uncertainty in attaining to this perfect adjustment, the following investigation was undertaken to ascertain the effect of any want of perfectness in the alignment. We shall also ascertain whether it be allowable to support the bar at the same points as it has been always supported, but upon *lever* rollers instead of upon rigid supports. In this case the four pressures would be equal, but the surfaces of the rollers not in one plane.

We shall have the more confidence in the results of the investigation inasmuch as we have found in the preceding section that the observed phenomena of flexure are in all but perfect accordance with computed results from theory.

1.

Let us suppose the bar to be resting, with unequal pressures, upon four supporting points or rollers, of which the two outer are in a horizontal straight line, but the two inner not in that line. It will be easily seen that by this we do not limit the generality of the results. In the figure let P Q P′ Q′ be the four points of support, the distances apart $Q'P'=P'P=PQ=b$; the whole



length of the bar $=a$. Also let

$$\begin{aligned}
\text{Pressure at } P &= P \\
\text{,, } \quad Q &= Q \\
\text{,, } \quad P' &= P' \\
\text{,, } \quad Q' &= Q'
\end{aligned}$$

Then if w be the weight of the bar

$$P + Q + P' + Q' = w \tag{1}$$

We assume further that the bar rests symmetrically upon the rollers, that is, that the centre of the bar is midway between the rollers P P′. Taking moments about the centre of the bar we get the equation

$$P + 3\,Q = P' + 3\,Q' \tag{2}$$

These two equations give

$$\begin{aligned} P &= \frac{w}{2} - 2\,Q + Q' \\ P' &= \frac{w}{2} + Q - 2\,Q' \end{aligned} \tag{3}$$

Now let

$$\begin{aligned} Q &= \frac{\lambda}{2} w \\ Q' &= \frac{\lambda'}{2} w \end{aligned} \tag{4}$$

$$\begin{aligned} \text{then, } P &= \frac{w}{2}(1 - 2\lambda + \lambda') \\ P' &= \frac{w}{2}(1 + \lambda - 2\lambda') \end{aligned} \tag{5}$$

The points Q Q′ being in a horizontal line, let the vertical ordinates of P P′ be $\delta\,\delta'$, that is their distances *above* the horizontal line. Taking Q Q′ for the axis of x

$$\begin{array}{lll} \text{the co-ordinates of Q are } & x = \frac{3}{2}b, & y = 0 \\ \quad ,, \qquad\qquad P \;\; ,, & x = \frac{1}{2}b, & y = \delta \\ \quad ,, \qquad\qquad P' \;\; ,, & x = -\frac{1}{2}b, & y = \delta' \\ \quad ,, \qquad\qquad Q' \;\; ,, & x = -\frac{3}{2}b, & y = 0 \\ \qquad \text{centre of bar} & x = 0 & y = \delta_0 \end{array} \tag{6}$$

Also let the inclination of the bar to the horizontal line

$$\begin{array}{ll} \text{or, } \dfrac{dy}{dx} \text{ at centre of bar} & = s \\ \quad ,, \quad \text{the point P} & = p \\ \quad ,, \qquad ,, \quad\;\; Q & = q \end{array} \tag{7}$$

Consider now the forces tending to bend the bar at any point between the centre and P. Let the distance of this point from the centre be x: the sum of the moments of the upward reactions of the supports is

$$P\left(\frac{b}{2} - x\right) + Q\left(\frac{3b}{2} - x\right)$$

And the sum of the moments of the parallel forces constituting the weight of that portion of the bar which is to the right of the point we are considering is

$$-\frac{w}{2a}\left(\frac{a}{2} - x\right)^2$$

The sum of the moments of the elastic forces brought into action at the section of the bar in question is by equation (4), page 22,

$$\frac{wak^2}{12\alpha\varrho}$$

Consequently

$$\frac{wak^2}{12\alpha\varrho} = (\mathrm{P} + 3\,\mathrm{Q})\frac{b}{2} - (\mathrm{P} + \mathrm{Q})\,x - \frac{wa}{8} + \frac{w}{2}\,x - \frac{w}{2a}\,x^2 \tag{8}$$

Now by (4) and (5)

$$\mathrm{P} + 3\,\mathrm{Q} = \frac{w}{2}(1 + \lambda + \lambda')$$

$$\mathrm{P} + \mathrm{Q} = \frac{w}{2}(1 - \lambda + \lambda')$$

Consequently by substitution (8) becomes

$$\frac{ak^2}{6\alpha\varrho} = (1 + \lambda + \lambda')\frac{b}{2} - (1 - \lambda + \lambda')\,x - \frac{a}{4} + x - \frac{x^2}{a} \tag{9}$$

or if as before $\frac{ak^2}{6\alpha} = \frac{1}{\mu}$

$$\frac{1}{\mu}\cdot\frac{d^2y}{dx^2} = (1 + \lambda + \lambda')\frac{b}{2} + (\lambda - \lambda')\,x - \frac{a}{4} - \frac{x^2}{a} \tag{10}$$

Integrating this equation

$$\frac{1}{\mu}\frac{dy}{dx} = \mathrm{C}_1 + \frac{b}{2}(1 + \lambda + \lambda')\,x + \frac{\lambda - \lambda'}{2}\,x^2 - \frac{a}{4}\,x - \frac{x^3}{3a}$$

Making $x = 0$

$$\frac{1}{\mu}\,s = \mathrm{C}_1$$

$$\therefore\quad \frac{1}{\mu}\frac{dy}{dx} = \frac{s}{\mu} + \left\{\frac{b}{2}(1 + \lambda + \lambda') - \frac{a}{4}\right\}x + \frac{\lambda - \lambda'}{2}\,x^2 - \frac{x^3}{3a} \tag{11}$$

Again, in this equation make $x = \frac{b}{2}$, and we have

$$\frac{p}{\mu} = \frac{b^2}{8}(2 + 3\lambda + \lambda') - \frac{ab}{8} - \frac{b^3}{24a} + \frac{s}{\mu} \tag{12}$$

Integrating a second time equation (11)

$$\frac{y}{\mu} = \mathrm{C}_2 + \frac{s}{\mu}\,x + \left\{\frac{b}{2}(1 + \lambda + \lambda') - \frac{a}{4}\right\}\frac{x^2}{2} + \frac{\lambda - \lambda'}{6}\,x^3 - \frac{x^4}{12a}$$

When $x = 0$, $y = \delta_0$ consequently

$$\frac{y}{\mu} = \frac{\delta_0}{\mu} + \frac{sx}{\mu} + \left\{\frac{b}{2}(1 + \lambda + \lambda') - \frac{a}{4}\right\}\frac{x^2}{2} + \frac{\lambda - \lambda'}{6}\,x^3 - \frac{x^4}{12a} \tag{13}$$

This is the equation of the curve of the neutral axis from P′ to P. If we make $x = \frac{b}{2}$ and therefore $y = \delta$

$$\frac{\delta}{\mu} = \frac{\delta_0}{\mu} + \frac{sb}{2\mu} + \frac{b^3}{48}(3 + 4\lambda + 2\lambda') - \frac{ab^2}{32} - \frac{b^4}{12\cdot 16a} \tag{14}$$

Again; take a point between P and Q whose abscissa is x: the sum of the moments of the forces tending to turn the right-hand part of the bar about this point is

$$Q\left(\tfrac{3}{2}b - x\right) - \frac{w}{2a}\left(\frac{a}{2} - x\right)^2$$

consequently for the equation of the curve

$$\frac{1}{\mu}\frac{d^2y}{dx^2} = \lambda\left(\tfrac{3}{2}b - x\right) - \frac{a}{4} + x - \frac{x^2}{a} \tag{15}$$

Integrating this

$$\frac{1}{\mu}\frac{dy}{dx} = C_3 + \left(\tfrac{3}{2}b\lambda - \frac{a}{4}\right)x + (1-\lambda)\frac{x^2}{2} - \frac{x^3}{3a}$$

making $x = \dfrac{b}{2}$

$$\frac{p}{\mu} = C_3 + \tfrac{5}{8}b^2\lambda - \frac{ab}{8} + \frac{b^2}{8} - \frac{b^3}{24a}$$

subtracting equation (12) from this we get

$$0 = C_3 - \frac{b^2}{8}(1 - 2\lambda + \lambda') - \frac{s}{\mu}$$

and therefore

$$\frac{1}{\mu}\frac{dy}{dx} = \frac{b^2}{8}(1 - 2\lambda + \lambda') + \frac{s}{\mu} + \left(\frac{3}{2}b\lambda - \frac{a}{4}\right)x + \frac{1-\lambda}{2}x^2 - \frac{x^3}{3a} \tag{16}$$

Here let x again $= \tfrac{3}{2}b$, and we get

$$\frac{q}{\mu} = \frac{s}{\mu} + \frac{b^2}{8}(10 + 7\lambda + \lambda') - \frac{3ab}{8} - \frac{9b^3}{8a} \tag{17}$$

Integrating once more equation (16)

$$\frac{y}{\mu} = C_4 + \frac{sx}{\mu} + \frac{b^2}{8}(1 - 2\lambda + \lambda')x + \left(\frac{3}{4}b\lambda - \frac{a}{8}\right)x^2 + \frac{1-\lambda}{6}x^3 - \frac{x^4}{12a} \tag{18}$$

which is the equation of the curve of the neutral axis from P to Q: but we have to determine the constant C_4.

If in this last equation $x = \dfrac{b}{2}$ then $y = \delta$, and we get after some reduction

$$\frac{\delta}{\mu} = \frac{sb}{2\mu} + \frac{b^3}{48}(4 + 2\lambda + 3\lambda') - \frac{ab^2}{32} - \frac{b^4}{12 \cdot 16a} + C_4$$

subtracting from this equation (14) the result is

$$0 = -\frac{\delta_0}{\mu} + \frac{b^3}{48}(1 - 2\lambda + \lambda') + C_4$$

and substituting this in (18) we have

$$\frac{y}{\mu} = \frac{\delta_0}{\mu} - \frac{b^3}{48}(1-2\lambda+\lambda') + \frac{sx}{\mu} + \frac{b^2}{8}(1-2\lambda+\lambda')x + \left(\frac{3}{4}b\lambda - \frac{a}{8}\right)x^2 + \frac{1-\lambda}{6}x^3 - \frac{x^4}{12a} \tag{19}$$

Now when $x = \frac{3}{2}b$, $y = 0$, consequently after a little reduction

$$0 = \frac{\delta_0}{\mu} + \frac{3bs}{2\mu} + \frac{b^3}{48}(35 + 38\lambda + 8\lambda') - \frac{9ab^2}{32} - \frac{27b^4}{64a} \qquad (20)$$

We arrive finally at the points to the right of Q. Here the equation of moments is

$$\frac{1}{\mu}\frac{d^2y}{dx^2} = -\frac{a}{4} + x - \frac{x^2}{a} \qquad (21)$$

$$\frac{1}{\mu}\frac{dy}{dx} = C_5 - \frac{a}{4}x + \frac{x^2}{2} - \frac{x^3}{3a}$$

making $x = \frac{3}{2}b$

$$\frac{q}{\mu} = C_5 - \frac{3ab}{8} + \frac{9b^2}{8} - \frac{9b^3}{8a}$$

Subtract from this equation (17) and we get

$$0 = C_5 - \frac{s}{\mu} - \frac{b^2}{8}(1 + 7\lambda + \lambda')$$

and substituting this value of C_5

$$\frac{1}{\mu}\frac{dy}{dx} = \frac{s}{\mu} + \frac{b^2}{8}(1 + 7\lambda + \lambda') - \frac{a}{4}x + \frac{x^2}{2} - \frac{x^3}{3a} \qquad (22)$$

Integrating this equation

$$\frac{y}{\mu} = C_6 + \frac{sx}{\mu} + \frac{b^2}{8}(1 + 7\lambda + \lambda')x - \frac{ax^2}{8} + \frac{x^3}{6} - \frac{x^4}{12a}$$

Make $x = \frac{3}{2}b$ and we have since here $y=0$

$$0 = C_6 + \frac{3sb}{2\mu} + \frac{b^3}{16}(3 + 21\lambda + 3\lambda') - \frac{9ab^2}{32} + \frac{9b^3}{16} - \frac{27b^4}{64a}$$

From this subtract equation (20) and we get

$$0 = C_6 + \frac{b^3}{48}(1 + 25\lambda + \lambda') - \frac{\delta_0}{\mu}$$

Substituting this value of C_6 we get finally for the equation of the neutral axis to the right of Q,

$$\frac{y}{\mu} = \frac{\delta_0}{\mu} - \frac{b^3}{48}(1 + 25\lambda + \lambda') + \frac{sx}{\mu} + \frac{b^2}{8}(1 + 7\lambda + \lambda')x - \frac{ax^2}{8} + \frac{x^3}{6} - \frac{x^4}{12a} \qquad (23)$$

We now for the sake of clearness bring together the equations of the three parts of the bar as follows:

$$\text{PP}: \frac{y}{\mu} = \frac{\delta_0}{\mu} + \frac{sx}{\mu} + \left\{\frac{b}{2}(1 + \lambda + \lambda') - \frac{a}{4}\right\}\frac{x^2}{2} + \frac{\lambda - \lambda'}{6}x^3 - \frac{x^4}{12a} \qquad \text{(A)}$$

$$\text{PQ}: \frac{y}{\mu} = \frac{\delta_0}{\mu} - \frac{b^3}{48}(1 - 2\lambda + \lambda') + \frac{sx}{\mu} + \frac{b^2}{8}(1 - 2\lambda + \lambda')x + \left(\frac{3}{4}b\lambda - \frac{a}{8}\right)x^2 + \frac{1 - \lambda}{6}x^3 - \frac{x^4}{12a} \qquad \text{(B)}$$

$$\text{Q}-: \frac{y}{\mu} = \frac{\delta_0}{\mu} - \frac{b^3}{48}(1 + 25\lambda + \lambda') + \frac{sx}{\mu} + \frac{b^2}{8}(1 + 7\lambda + \lambda')x - \frac{ax^2}{8} + \frac{x^3}{6} - \frac{x^4}{12a} \qquad \text{(C)}$$

We have now to find the eight quantities $\delta_0\ s\ p\ q\ p'\ q'\ \lambda\ \lambda'$ in terms of $\delta\ \delta'$. For this purpose we shall make use of equations (12), (17), (14), (20), and those corresponding to them for the left-hand part of the bar. In order to obtain these it is necessary in each of the above to alter λ into λ', λ' into λ, and s into $-s$. Thus we have the following :—

$$\frac{p}{\mu} = \frac{s}{\mu} + \frac{b^2}{8}(2 + 3\lambda + \lambda') - \frac{ab}{8} - \frac{b^3}{24a} \tag{24}$$

$$\frac{p'}{\mu} = -\frac{s}{\mu} + \frac{b^2}{8}(2 + 3\lambda' + \lambda) - \frac{ab}{8} - \frac{b^3}{24a} \tag{25}$$

$$\frac{q}{\mu} = \frac{s}{\mu} + \frac{b^2}{8}(10 + 7\lambda + \lambda') - \frac{3ab}{8} - \frac{9b^3}{8a} \tag{26}$$

$$\frac{q'}{\mu} = -\frac{s}{\mu} + \frac{b^2}{8}(10 + 7\lambda' + \lambda) - \frac{3ab}{8} - \frac{9b^3}{8a} \tag{27}$$

$$\frac{\delta}{\mu} = \frac{\delta_0}{\mu} + \frac{sb}{2\mu} + \frac{b^3}{48}(3 + 4\lambda + 2\lambda') - \frac{ab^2}{32} - \frac{b^4}{12\cdot 16a} \tag{28}$$

$$\frac{\delta'}{\mu} = \frac{\delta_0}{\mu} - \frac{sb}{2\mu} + \frac{b^3}{48}(3 + 4\lambda' + 2\lambda) - \frac{ab^2}{32} - \frac{b^4}{12\cdot 16a} \tag{29}$$

$$0 = \frac{\delta_0}{\mu} + \frac{3bs}{2\mu} + \frac{b^3}{48}(35 + 38\lambda + 8\lambda') - \frac{9ab^2}{32} - \frac{27b^4}{64a} \tag{30}$$

$$0 = \frac{\delta_0}{\mu} - \frac{3bs}{2\mu} + \frac{b^3}{48}(35 + 38\lambda' + 8\lambda) - \frac{9ab^2}{32} - \frac{27b^4}{64a} \tag{31}$$

From the last four equations, performing the operations (28)—$\frac{2}{3}$ (30)—$\frac{1}{3}$ (31) and (29)—$\frac{1}{3}$ (30)—$\frac{2}{3}$ (31) we get—

$$\frac{\delta}{\mu} = \frac{b^3}{6}(-4 - 3\lambda - 2\lambda') + \frac{ab^2}{4} + \frac{5}{12}\frac{b^4}{a} \tag{32}$$

$$\frac{\delta'}{\mu} = \frac{b^3}{6}(-4 - 3\lambda' - 2\lambda) + \frac{ab^2}{4} + \frac{5}{12}\frac{b^4}{a} \tag{33}$$

and from these—

$$\lambda = \frac{3}{10}\frac{a}{b} + \frac{5}{10}\frac{b}{a} - \frac{8}{10} + \frac{6}{5}\frac{2\delta' - 3\delta}{\mu b^3} \tag{34}$$

$$\lambda' = \frac{3}{10}\frac{a}{b} + \frac{5}{10}\frac{b}{a} - \frac{8}{10} + \frac{6}{5}\frac{2\delta - 3\delta'}{\mu b^3} \tag{35}$$

Again, from $\frac{1}{2}$ (30) + $\frac{1}{2}$ (31), with (34) + (35)

$$\frac{\delta_0}{\mu} = \frac{b^3}{16}\left(\frac{3}{5} - \frac{1}{10}\frac{a}{b} - \frac{11}{12}\frac{b}{a}\right) + \frac{23}{40}\frac{\delta + \delta'}{\mu} \tag{36}$$

The difference of (30) and (31) gives

$$s = \frac{5}{4}\cdot\frac{\delta - \delta'}{b} \tag{37}$$

We can now easily obtain $p\,p'\,q\,q'$ in terms of $\delta\,\delta'$, but as they are not required for our purpose, we shall write down now the equations of the curves into which the right-hand part of the bar is bent in terms of $\delta\,\delta'$. The equations (A) (B) (C) by substituting the values of $\lambda\,\lambda'\,\delta_0\,s$ become as follows :

1st, the equation of the part P P′ :—

$$\frac{y}{\mu}=\frac{b^3}{16}\left(\tfrac{3}{5}-\tfrac{1}{10}\frac{a}{b}-\tfrac{11}{12}\frac{b}{a}\right)+\tfrac{23}{40}\,\frac{\delta+\delta'}{\mu}+\tfrac{5}{4}\,\frac{\delta-\delta'}{\mu\,b}x$$

$$+\left\{\frac{b}{40}\left(-6+\frac{a}{b}+10\frac{b}{a}\right)-\tfrac{3}{10}\,\frac{\delta+\delta'}{\mu\,b^2}\right\}x^2-\frac{\delta-\delta'}{\mu\,b^3}x^3-\frac{x^4}{12a} \qquad (\text{A}')$$

2d, the equation of P Q :

$$\frac{y}{\mu}=-\tfrac{3}{64}\,\frac{b^4}{a}+\tfrac{15}{40}\,\frac{\delta+2\delta'}{\mu}+\left\{\frac{b^2}{80}\left(18-3\frac{a}{b}-5\frac{a}{b}\right)+\frac{49\delta-46\delta'}{20\,\mu\,b}\right\}x$$

$$+\left\{\frac{b}{40}\left(-24+4\frac{a}{b}+15\frac{b}{a}\right)+\tfrac{9}{10}\,\frac{-3\delta+2\delta'}{\mu\,b^2}\right\}x^2$$

$$+\left\{\tfrac{3}{10}-\tfrac{1}{20}\,\frac{a}{b}-\tfrac{1}{12}\frac{b}{a}-\tfrac{1}{5}\,\frac{-3\delta+2\delta'}{\mu\,b^3}\right\}x^3-\frac{x^4}{12a} \qquad (\text{B}')$$

3d, the equation of the part of bar beyond Q :

$$\frac{y}{\mu}=\frac{3b^3}{16}\left(\tfrac{12}{5}-\tfrac{9}{10}\frac{a}{b}-\tfrac{7}{4}\frac{b}{a}\right)+\tfrac{3}{5}\cdot\frac{4\delta-\delta'}{\mu}$$

$$+\left\{\frac{b^2}{8}\left(-\tfrac{27}{5}+\tfrac{12}{5}\frac{a}{b}+4\frac{b}{a}\right)-\tfrac{2}{5}\,\frac{4\delta-\delta'}{\mu\,b}\right\}x-\frac{a}{8}x^2+\tfrac{1}{6}x^3-\frac{x^4}{12a} \qquad (\text{C}')$$

2.

The last equations written down express completely the form of the bar, and the equations (34) (35) give us the pressures upon the supports when $\delta\,\delta'$ are given. For the four pressures we have from equations (4), (5), (34), (35) :

$$\frac{\text{P}}{w}=\quad\tfrac{18}{20}-\tfrac{3}{20}\frac{a}{b}-\tfrac{5}{20}\frac{b}{a}+\tfrac{6}{10}\cdot\frac{8\,\delta-7\delta'}{\mu\,b^3} \qquad (38)$$

$$\frac{\text{P}'}{w}=\quad\tfrac{18}{20}-\tfrac{3}{20}\frac{a}{b}-\tfrac{5}{20}\frac{b}{a}+\tfrac{6}{10}\cdot\frac{-7\delta+8\delta'}{\mu\,b^3} \qquad (39)$$

$$\frac{\text{Q}}{w}=-\tfrac{8}{20}+\tfrac{3}{20}\frac{a}{b}+\tfrac{5}{20}\frac{b}{a}+\tfrac{6}{10}\cdot\frac{-3\delta+2\delta'}{\mu\,b^3} \qquad (40)$$

$$\frac{\text{Q}'}{w}=-\tfrac{8}{20}+\tfrac{3}{20}\frac{a}{b}+\tfrac{5}{20}\frac{b}{a}+\tfrac{6}{10}\cdot\frac{2\delta-3\delta'}{\mu\,b^3} \qquad (41)$$

We shall now in these formulæ substitute the values of a, b, and μ, our unit of length being one inch. We shall take for the weight of a cubic foot of cast steel 490 lbs., and the modulus of elasticity 30,000,000. If h, k be the breadth and depth of the bar, its weight

$$w=\frac{490}{12^3}ahk$$

This applied as a direct force of compression to the bar would compress it the amount

$$\frac{w}{30{,}000{,}000}\cdot a\cdot\frac{1}{hk}=\frac{49a^2}{3\cdot(1200)^3}=\alpha$$

Consequently

$$\frac{1}{\mu} = \frac{ak^2}{6\alpha} = \frac{k^2(1200)^3}{98\,a}$$

But $k = 0 \cdot 39$; $a = 75 \cdot 5$;

$$\therefore \frac{1}{\mu} = \frac{1200}{49} \cdot \frac{(468)^2}{151} = 35522$$

$$\mu = \cdot 000028151$$

Substituting these values in (A′) (B′) (C′), we have the following :—

$$100\,y = -\cdot 0192 + 57 \cdot 5\,\delta + 57 \cdot 5\,\delta' + (58 \cdot 140\,\delta - 58 \cdot 140\,\delta')\,\frac{x}{10}$$
$$+ (\cdot 0544 - 6 \cdot 490\,\delta - 6 \cdot 49\,\delta')\left(\frac{x}{10}\right)^2 + (-10 \cdot 062\,\delta + 10 \cdot 062\,\delta')\left(\frac{x}{10}\right)^3$$
$$-\cdot 031072\left(\frac{x}{10}\right)^4 \qquad (A_1)$$

$$100\,y = -\cdot 3735 + 37 \cdot 5\,\delta + 75 \cdot 0\,\delta' + (\cdot 9827 + 113 \cdot 95\,\delta - 106 \cdot 98\,\delta')\left(\frac{x}{10}\right)$$
$$+ (-\cdot 8598 - 58 \cdot 41\,\delta + 38 \cdot 94\,\delta')\left(\frac{x}{10}\right)^2 + (\cdot 2834 + 6 \cdot 037\,\delta - 4 \cdot 025\,\delta')\left(\frac{x}{10}\right)^3$$
$$-\cdot 031072\left(\frac{x}{10}\right)^4 \qquad (B_1)$$

$$100\,y = -6 \cdot 6035 + 240\,\delta - 60\,\delta' + (6 \cdot 7781 - 74 \cdot 419\,\delta + 18 \cdot 605\,\delta')\left(\frac{x}{10}\right)$$
$$-2 \cdot 6568\left(\frac{x}{10}\right)^2 + \cdot 46919\left(\frac{x}{10}\right)^3 - \cdot 031072\left(\frac{x}{10}\right)^4 \qquad (C_1)$$

$$\frac{P}{\omega} = \cdot 3021 + 17 \cdot 156\,\delta - 15 \cdot 012\,\delta' \qquad (42)$$

$$\frac{P'}{\omega} = \cdot 3021 - 15 \cdot 012\,\delta + 17 \cdot 156\,\delta' \qquad (43)$$

$$\frac{Q}{\omega} = \cdot 1979 - 6 \cdot 434\,\delta + 4 \cdot 289\,\delta' \qquad (44)$$

$$\frac{Q'}{\omega} = \cdot 1979 + 4 \cdot 289\,\delta - 6 \cdot 434\,\delta' \qquad (45)$$

The sum of these last equations being as it should be

$$\frac{P}{w} + \frac{P'}{w} + \frac{Q}{w} + \frac{Q'}{w} = 1$$

3.

In the expressions for the reaction of the supports first given, the symbols δ δ' represent the height of the inner supports above the line joining the outer supports expressed in inches. But it will be more convenient to express these quantities in hundredths of inches, that is, making $\frac{1}{100}$ of an inch the *unit*. The equations will then stand thus:

$$\frac{P}{w} = \cdot 3021 + \cdot 17156\,\delta - \cdot 15012\,\delta' \tag{46}$$

$$\frac{P'}{w} = \cdot 3021 - \cdot 15012\,\delta + \cdot 17156\,\delta' \tag{47}$$

$$\frac{Q}{w} = \cdot 1979 - \cdot 06434\,\delta + \cdot 04289\,\delta' \tag{48}$$

$$\frac{Q'}{w} = \cdot 1979 + \cdot 04289\,\delta - \cdot 06434\,\delta' \tag{49}$$

Now in order that the bar may be resting on all four rollers it is necessary that none of the quantities P P′ Q Q′ should be negative. If for certain values given to δ and δ', one or more of the reactions turned out to be negative, it would show that the bar required to be drawn or pressed down to bring it into contact with all the rollers. The values of δ δ' are then restricted within certain limits which must be ascertained. We shall proceed in the following manner:

Multiply and divide the equations (46) (47) by $\sqrt{(\cdot 17156)^2 + (\cdot 15012)^2}$, and multiply and divide (48) (49) by $\sqrt{(\cdot 06434)^2 + (\cdot 04289)^2}$, writing them thus:

$$\frac{P}{w} = 0 \cdot 2280\,(1 \cdot 3252 + \cdot 7526\,\delta - \cdot 6585\,\delta') \tag{50}$$

$$\frac{P'}{w} = 0 \cdot 2280\,(1 \cdot 3252 - \cdot 6585\,\delta + \cdot 7526\,\delta') \tag{51}$$

$$\frac{Q}{w} = 0 \cdot 0773\,(2 \cdot 5593 - \cdot 8321\,\delta + \cdot 5547\,\delta') \tag{52}$$

$$\frac{Q'}{w} = 0 \cdot 0773\,(2 \cdot 5593 + \cdot 5547\,\delta - \cdot 8321\,\delta') \tag{53}$$

If $x \cos \beta + y \sin \beta - \alpha$ be the equation of a straight line, α the perpendicular from the origin of co-ordinates on that line, always a positive quantity, β increasing from 0 to 360°; the perpendicular from a point whose co-ordinates are

$$x = \delta, y = \delta'$$

upon the line, is, if the point be on the *same* side of the line with the origin,

$$\alpha - \delta \cos \beta - \delta' \sin \beta$$

Now the quantities within the brackets in (50), (51), (52), (53), are of this form, the values of α and β being:—

$$\alpha_1 = 1 \cdot 3252 \quad \beta_1 = 138^\circ\ 49' \tag{50'}$$

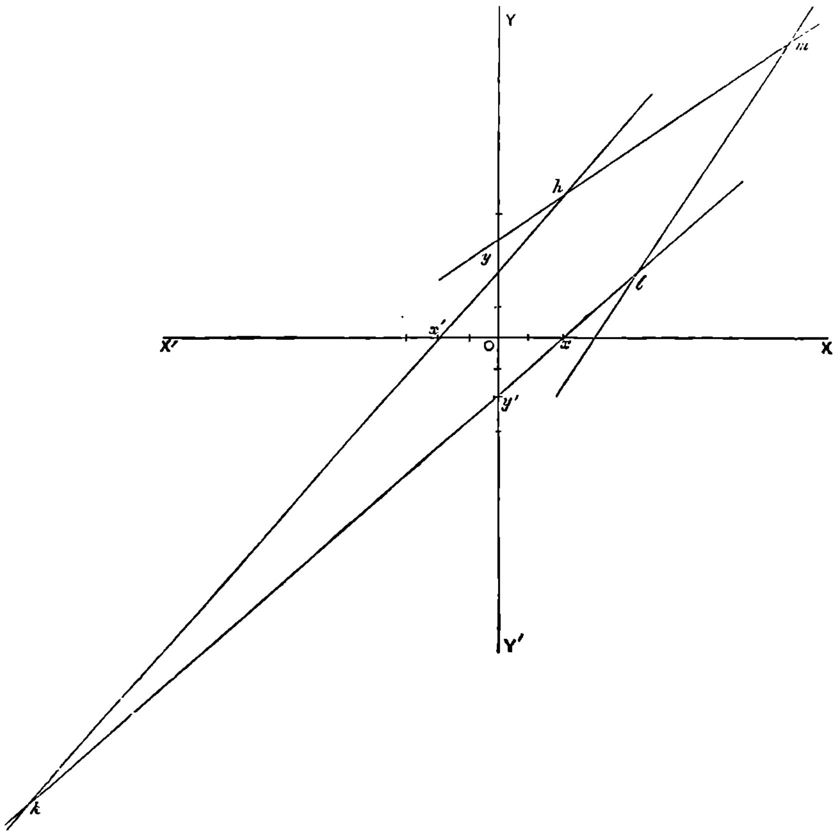
$$\alpha_2 = 1 \cdot 3252 \quad \beta_2 = 311^\circ\ 11' \tag{51'}$$

$$\alpha_3 = 2 \cdot 5593 \quad \beta_3 = 326^\circ\ 19' \tag{52'}$$

$$\alpha_4 = 2 \cdot 5593 \quad \beta_4 = 123^\circ\ 41' \tag{53'}$$

which determine four straight lines, and these straight lines fix the limits within which δ δ' must be confined in order that the bar may take a bearing on each of the rollers.

In the adjoining figure let X′ X be the axis of x, Y Y′ that of y intersecting in a right angle at the origin O. Construct the line $h\ k$ such that the perpendicular let fall upon it from the origin = 1·325, making an angle of 138° 49′ with O X. Draw $k\ l$ at the



same distance from the origin, but the perpendicular making an angle of 311° 11′ with O X. It will be seen that the point k in which these two lines intersect lies on the line bisecting the angle X′ O Y′. Again, construct the line $l\ m$, having its perpendicular = 2·559, and making an angle of 326° 19′ with O X; finally at the same perpendicular distance from the origin, the line $m\ h$, the perpendicular making an angle of 123° 41′ with O X. The four lines $h\ k$, $k\ l$, $l\ m$, $m\ h$, it will be seen form a quadrilateral which is symmetrically bisected by the line bisecting the angle X O Y.

Now take any point within the quadrilateral $h\ k\ l\ m$, and let the co-ordinates of that point be $x = \delta$, $y = \delta'$, and let the perpendiculars on the four lines be $p\ p'\ q\ q'$, then

$$\frac{P}{w} = \cdot 2280\, p$$

$$\frac{P'}{w} = \cdot 2280\, p'$$

$$\frac{Q}{w} = \cdot 0773\, q$$

$$\frac{Q'}{w} = \cdot 0773\, q'$$

We see now that $\delta\ \delta'$ are so limited that the point they represent must fall ***within*** the rectangle $h\ k\ l\ m$; for if we take a point without the rectangle, one or more of the perpendiculars will be negative, and therefore the pressures negative. We proceed to the following deductions: (1) As long as δ and δ' are of the same sign they may be comparatively large quantities, whether the sign be + or —, but larger if both are negative. If they are equal and at the *positive* maximum, (52) or (53) gives

$$2 \cdot 5593 - 0 \cdot 2774\ \delta = 0$$

$$\therefore \delta = 9 \cdot 22$$

that is, ·0922 inch. If both be equal and at the negative maximum, (50) or (51) gives

$$1 \cdot 3252 + 0 \cdot 0941\ \delta = 0$$

$$\therefore \delta = -14 \cdot 08$$

which is nearly a *seventh of an inch*.

In the former case there is no pressure on the supports Q Q′, in the latter there is no pressure on P P′.

(2) If the quantities $\delta\ \delta'$ are of opposite signs, then their range is much more limited, they must be such that the points they represent fall within either of the spaces X′ O Y′ or Y O X. If one be $+\frac{1}{100}$ and the other $-\frac{1}{100}$ of an inch, there will be *no contact* with one or other of P, P′.

(3) Suppose that all the supports being first in a horizontal line (that is, $\delta = 0$, $\delta' = 0$) the support P is *raised* until the pressure on P′ becomes zero, that is, until the bar is about to cease having contact with P′; then Ox represents the quantity δ by which P may be raised. Similarly supposing all the supports to be at first in a straight line, Ox' will represent the amount $(-\delta)$ by which P may be *lowered* without coming away from the bar. The values in hundredths of an inch are

$$Ox = 2 \cdot 01,\quad Ox' = -1 \cdot 76$$

Oy, Oy' have the same meaning with reference to the support P′.

(4) Suppose the bar to be resting only on Q′ and P, having bare contact with P′ and Q. The values of $\delta\ \delta'$ will be those which correspond to the point l, both positive, namely,

$$\delta = 4 \cdot 56$$

$$\delta' = 2 \cdot 23$$

and the pressure on Q′ and P : equations (38), (39), (40), (41).

$$\frac{P}{\omega} = \frac{3}{4}$$

$$\frac{Q'}{\omega} = \frac{1}{4}$$

(5) If the supports be truly in line, then we see by equations (42), (43), (44), (45) that

$$P = P' : Q = Q' :: 3 : 2$$

very nearly.

(6) If the pressures on the four rollers be equal, we must have

$$\delta = \delta' = -2 \cdot 43$$

hundredths of an inch. In this case by equation (A_1) we can show, making $x = 0$, that

$$\delta_0 = -2 \cdot 81$$

If therefore the Prussian bar were supported at the same four points of its length at which it has always been supported, but on lever rollers instead of rigid supports adjusted into a straight line, the centre of the bar would be deflected ·028 inch below those parts which were in contact with the outer rollers.

From what we have now seen we conclude that if the errors of adjustment δ δ' of the two centre rollers be confined to small quantities such as $\frac{1}{150}$ of an inch, there will not fail to be contact on all four rollers. If we knew only for certain that neither δ nor δ' exceed $\frac{1}{100}$, then we cannot say positively that there *will* be contact on all the rollers, but the chances are in favour of it.

4.

We shall next ascertain how much the horizontal projection of the bent bar differs from its unbent length. Let the equation of the curve be

$$y = A_0 + A_1 x + A_2 \frac{x^2}{2} + A_3 \frac{x^3}{3} + A_4 \frac{x^4}{4} \tag{54}$$

We require to find the difference of the length of this curve from $x = h$ to $x = k$, and its projection on the axis of x between the same limits. The character of the curve being such that $\frac{dy}{dx}$ is a very small quantity, we have

$$s = \int_h^k \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} dx = k - h + \frac{1}{2}\int_h^k \left(\frac{dy}{dx}\right)^2 dx$$

The difference in question therefore is

$$\sigma = \frac{1}{2}\int_h^k \left(\frac{dy}{dx}\right)^2 dx \tag{55}$$

differentiating (54) and squaring

$$\left(\frac{dy}{dx}\right)^2 = A_1^2 + 2\,A_1 A_2 x + (A_2^2 + 2\,A_1 A_3)\,x^2 + 2\,(A_1 A_4 + A_2 A_3)\,x^3$$
$$+ (2\,A_2 A_4 + A_3^2)\,x^4 + 2\,A_3 A_4 x^5 + A_4^2 x^6$$

Wherefore

$$\sigma = B_1 (k - h) + B_2 (k^2 - h^2) + B_3 (k^3 - h^3) + B_4 (k^4 - h^4) + B_5 (k^5 - h^5)$$
$$+ B_6 (k^6 - h^6) + B_7 (k^7 - h^7) \tag{56}$$

the values of B_1 B_2 .... being

$$B_1 = \tfrac{1}{2} A_1^2; \qquad B_2 = \tfrac{1}{2} A_1 A_2$$
$$B_3 = \tfrac{1}{6} (A_2^2 + 2\,A_1 A_3); \quad B_4 = \tfrac{1}{4} (A_1 A_4 + A_2 A_3); \quad B_5 = \tfrac{1}{10} (A_3^2 + 2\,A_2 A_4)$$
$$B_6 = \tfrac{1}{6} A_3 A_4 \qquad B_7 = \tfrac{1}{14} A_4^2$$

From the equations (A_1), (B_1), (C_1), page 118, we may, without any further difficulty than some little labour, obtain the values of B_1 B_2 ..... B_7 for each of the curves. In the equation of P P′ to obtain σ we shall integrate from $-\frac{1}{2}b$ to $+\frac{1}{2}b$. In the equation of the part P Q we shall integrate from $\frac{1}{2}b$ to $\frac{3}{2}b$.. For the part beyond Q we shall integrate from $\frac{3}{2}b$ to $\frac{1}{2}a$. For the part P′Q′ the result obtained for P Q will hold good on δ, δ' being interchanged; and for the part to the left of Q′ the result obtained for the part to the right of Q will hold good on $\delta\ \delta'$ being interchanged.

The results obtained for the five different parts are as follows:

$$\begin{aligned}
P'P;\ & +\cdot000000008 && -\cdot00000242\,\delta && -\cdot00000242\,\delta' \\
& +\cdot025116277\,\delta^2 && -\cdot04744185\,\delta\delta' && +\cdot02511627\,\delta'^2 \\
PQ;\ & +\cdot000000167 && -\cdot00003672\,\delta && +\cdot00002454\,\delta' \\
& +\cdot02995612\,\delta^2 && -\cdot00893287\,\delta\delta' && +\cdot00297677\,\delta'^2 \\
P'Q';\ & +\cdot000000167 && +\cdot00002454\,\delta && -\cdot00003672\,\delta' \\
& +\cdot00297677\,\delta^2 && -\cdot00893287\,\delta\delta' && +\cdot02995612\,\delta'^2 \\
Q\ldots;\ & +\cdot000000026 && -\cdot00003971\,\delta && +\cdot00000992\,\delta' \\
& +\cdot01522985\,\delta^2 && -\cdot00761493\,\delta\delta' && +\cdot00095186\,\delta'^2 \\
\ldots Q';\ & +\cdot000000026 && +\cdot00000992\,\delta && -\cdot00003971\,\delta' \\
& +\cdot000951866\,\delta^2 && -\cdot00761493\,\delta\delta' && +\cdot01522985\,\delta'^2
\end{aligned}$$

Adding together these five quantities we get the quantity σ for the whole bar as follows:

$$\sigma = \cdot00000039 - \cdot0000444\,(\delta + \delta') + \cdot0742\,\delta^2 - \cdot0805\,\delta\delta' + \cdot0742\,\delta'^2$$

This is the difference between the actual length of the bar and the horizontal projection of the same: $\delta\ \delta'$ being expressed in inches. If $\delta\ \delta'$ be expressed in hundredths of an inch;

$$1000000\,\sigma = 0\cdot39 - 0\cdot444\,(\delta + \delta') + 7\cdot42\,\delta^2 - 8\cdot05\,\delta\delta' + 7\cdot42\,\delta'^2$$

To apply this in some particular cases

(1) Let $\delta = 0$, $\delta' = 1\cdot00$,

$$\sigma = \cdot0000074 \text{ inch}$$

The probable error of a single observation (reading) with the micrometers in use is just $\pm\, 0\cdot000007$ inch.

(2) Let $\delta = \delta' = 2\cdot0$ then

$$\sigma = \cdot0000258 \text{ inch}$$

but if $\delta = \delta' = -\,2\cdot0$

$$\sigma = \cdot0000293 \text{ inch}$$

(3) If the two centre rollers be lowered until the pressures on all four are equal, we have seen that in this case $\delta = \delta' = -\,2\cdot43$, and corresponding shortening is

$$\sigma = \cdot0000426 \text{ inch}$$

which is equivalent to one and a half micrometer divisions.

We conclude from these results that if the rollers P P′ can be adjusted into line with an error not exceeding say the one hundred and fiftieth part of an inch, there need be no fear either of a false bearing or of an alteration of apparent length. If the bar were supported at the same points of contact as at present, but upon lever rollers, its length would differ very sensibly from the truth.

5.

We have seen that in the positions in which the supports of the Prussian Toise are actually placed, the pressures upon the centre supports are half as great again as the pressures on the outer supports. It may be interesting to inquire at what intervals the supports should have been placed so as to sustain equal pressures.

For this purpose in the equations (38), (39), (40), (41), make $\delta = \delta' = 0$: then since $P = P' = Q = Q'$ we must have

$$3\frac{a}{b} + 5\frac{b}{a} - 13 = 0$$

Solving this quadratic equation we get

$$\frac{b}{a} = 0 \cdot 256$$

Which, a being $= 75 \cdot 5$, gives

$$b = 19 \cdot 33 \text{ inches.}$$

By this arrangement the overhanging extremities of the bar would have been each 8·75 inches instead of, as they are, 5·5 inches.

XI.

DETERMINATION OF THE LENGTH OF THE PRUSSIAN TOISE.

1.

The comparisons of the Prussian Toise No. 10 with the Ordnance Toise are divided into six series, as follows :—

Series 1.	Comparisons on	September	10th, 11th, 12th, 14th; ten comparisons.
2.	"	"	18th, 19th, 21st, 22d, 23d; ten comparisons.
3.	"	"	29th, 30th, October 1st, 2d; ten comparisons.
4.	"	October	20th, 21st, 22d, 23d; ten comparisons.
5.	"	January	5th, 6th, 7th, 8th, 9th; fifteen comparisons.
6.	"	May	12th, 13th, 14th; ten comparisons.

In all 65 comparisons, that is, sixty-five visits to the bars, the readings at one visit being taken and entered precisely in the manner explained in the comparisons of **OT** and $\mathbf{Y}_{55}$. The total number of micrometer readings is $65 \times 36 = 2340$. If we consider that at each visit *three* comparisons are made, this would make the total number of comparisons, 195.

The bars were visited generally three times during the day, at hours as far apart as practicable. At the close of each visit the *contacts* were re-made, that is, the small contact pieces were drawn away from the ends of the Prussian Toise and brought up again to contact; thus for every visit there was a fresh contact made: the object being the elimination of any constant error from this source. Also at the close of each visit the bar left under the microscopes (alternately one and the other) was re-adjusted to focus. The space to be measured by each microscope in this operation being about 260 divisions —a large quantity, it was very necessary to guard against any constant error that might arise from focussing, although, indeed, it would appear from what has been shown at page 63 that very little error is to be feared in measuring such a quantity as 260 divisions. From these precautions it must be pretty clear that the errors arising from errors of contact and errors of focussing are fully brought out in our individual comparisons, and in great measure eliminated in the final result.

With respect to temperature, the first four series and the last series are at temperatures ranging between 58°.33 and 63°.69; and the fifth series at low temperature between 30°.79 and 34°.61. By this disposition we have ascertained the difference of length very nearly at the normal temperature, and have ascertained very accurately the difference of the expansions of the bars.

In general, the readings of the thermometers at the close of a visit are higher than the readings at the commencement. The reason is obvious,—the lighting of the candles and the warmth of the person of the observer conspiring to raise the temperature. But the actual amount is very small: from an examination of all the readings it appears that the *average* increase is 0°.02.

After each series of ten comparisons the Prussian Toise was dismounted and placed in its case while other comparisons were made on other bars. Thus each series of ten is entirely independent of the other. Apparently, there can be no error common to any two, as *all* adjustments are completely thrown out and renewed. An important point for consideration is the proper *support* of the toise. As has been explained, it rests on four rollers, which are supposed to be in a straight line, that is, their upper surfaces to have a common tangent plane. In order to attain to this, the upper surfaces were brought into line by means of a long level, mounted on a piece of very carefully planed mahogany about 2 feet long: the value of one division in this level is 4″. The level is first laid on the left roller and the left centre roller, and, after a minute or two, is read. It is then placed on the two centre rollers and read. Finally, on the right centre roller and the right roller, and read. From these readings it is easy to see which of the centre rollers is too high or too low with respect to the extreme rollers (which are fixed and do not admit of vertical motion). The centre rollers being accordingly elevated or depressed, the level readings in the three positions are again taken. If, as is very probable, one or both of the centre rollers require further adjustment, they are altered and the level read again. This process is continued until the readings of the level in its three positions are very nearly the same, that is, within one or two divisions. The distance of a contiguous pair of rollers is 21·5 inches, consequently one division of the level will indicate a vertical displacement of $21\cdot5 \tan 4'' = \cdot00042$ inch. In the first series of comparisons the residual readings of the level were not recorded. On removing the toise at the close of the second series the rollers were tested, and it was found that the three rollers on the right were sensibly in line, the left roller indicated 5 divisions of the level in error which is about $\frac{1}{500}$ of an inch. At the commencement of the third series, after the levelling of the rollers, the residual readings of the level in the three positions were

15·0, 16·0, 17·5

which indicate very minute vertical displacements of rollers.

At the commencement of the fourth series, after the levelling of the rollers the residual readings were

15·5, 16·5, 16·5

At the commencement of the fifth

17·0 17·0 18·3

At the commencement of the sixth

16·0 16·7 17·5

From this it would appear that there is not much to fear from any curvature of the toise. At any rate the error is not the same for any two series.

With respect to the form of the rollers, the intention was that *one* should be a perfect cylinder with its axis truly horizontal, and the other three barrel-shaped. Thus the contacts of the Prussian Toise would be along its whole breadth on the cylinder roller, but merely in a point on the barrel-shaped rollers, and that point at the centre of the breadth. The cylinder roller to be one of the centre ones. But it was not discovered at first that all the rollers were truly cylinders. This was remedied after the first series. As it would be difficult or impossible to adjust these rollers so that their four axes should be truly parallel, so we must expect that the toise resting on them would be supported on its outer edges, and so subjected to a slight degree of torsion. This circumstance may, perhaps, render the first series of ten comparisons less unexceptionable than the remainder.

The contact was made at either end of the Prussian Toise, as near as practicable to the centre of the small circular polished disk forming the *end of the toise*. There is *near*, but not *at*, the centre of each of these disks a small speck, visible with a magnifying glass, probably of rust. In making the contacts care was taken to avoid these spots, still keeping as near as possible to the centre.

The following Table contains the results of these comparisons; each line shows the mean result of one visit. T_{10} denotes the length of the Prussian Toise, T_0 that of the Ordnance Toise, σ the space on the contact apparatus.

RESULTS OF COMPARISON OF PRUSSIAN TOISE WITH ORDNANCE TOISE.

Date.	Mean Temp.	Prussian Toise.	Ordnance Toise.	Difference in Micrometer Divisions.	$T_{10} - T_0 + \sigma$
1863:					
Sept. 10	63·69°	−195·74 h + 3·67 k	+199·90 h + 122·96 k	+395·64 h + 119·29 k	409·70
„ „	63·67	−192·11 h − 0·91 k	+190·47 h + 132·53 k	+382·58 h + 133·44 k	410·62
„ 11	62·85	−192·73 h + 9·71 k	+194·92 h + 136·44 k	+387·65 h + 126·73 k	409·29
„ „	63·02	− 62·98 h − 121·11 k	+164·38 h + 165·44 k	+227·36 h + 286·55 k	409·41
„ 12	62·08	− 48·89 h − 123·73 k	+174·43 h + 168·58 k	+223·32 h + 292.31 k	410·80
„ „	62·11	− 53·27 h − 120·51 k	+179·19 h + 162·13 k	+232·46 h + 282·64 k	410·34
„ „	62·34	− 40·41 h − 127·77 k	+216·03 h + 130·24 k	+256·44 h + 258·01 k	409·74
„ 14	61·22	− 50·27 h − 103·93 k	+224·63 h + 135·14 k	+274·90 h + 239·07 k	409·31
„ „	61·33	− 57·79 h − 98·44 k	+209·69 h + 148·87 k	+267·48 h + 247·31 k	409·99
„ „	61·37	− 67·92 h − 87·26 k	+199·12 h + 160·17 k	+267·04 h + 247·43 k	409·73
„ 18	61·73	−298·28 h + 155·38 k	+199·47 h + 176·49 k	+497·75 h + 21·11 k	412·53
„ 19	61·60	−264·33 h + 124·56 k	+190·53 h + 186·89 k	+454·86 h + 62·33 k	411·33
„ „	62·26	− 81·51 h − 65·30 k	+172·16 h + 200·02 k	+253·67 h + 265·32 k	413·38
„ 21	61·59	− 81·04 h − 58·39 k	+185·79 h + 192·69 k	+266·83 h + 251·08 k	412·47
„ „	61·67	− 70·16 h − 71·17 k	+191·13 h + 185·09 k	+261·29 h + 256·26 k	412·21
„ „	61·96	− 92·91 h − 51·81 k	+175·78 h + 197·43 k	+268·69 h + 249·24 k	412·49
„ 22	60·94	− 84·60 h − 47·54 k	+185·72 h + 198·32 k	+270·32 h + 245·86 k	411·09
„ „	61·01	− 90·69 h − 42·52 k	+183·96 h + 200·43 k	+274·65 h + 242·95 k	412·21
„ „	61·01	− 74·40 h − 59·18 k	+192·43 h + 192 51 k	+266·83 h + 251·69 k	412·96
„ 23	60·30	− 69·69 h − 58·08 k	+192·90 h + 197·18 k	+262·59 h + 255·26 k	412·44
„ 29	59·18	−136·14 h − 56·63 k	+ 94·32 h + 228·67 k	+230·46 h + 285·30 k	410·87
„ „	59·14	−154·98 h − 36·71 k	+115·57 h + 209·42 k	+270·55 h + 246·13 k	411·48
„ „	59·07	−168·82 h − 20·20 k	+ 80·68 h + 247·86 k	+249·50 h + 268·06 k	412·25
„ 30	58·42	−144·61 h − 39·22 k	+139·19 h + 193·42 k	+283·80 h + 232·64 k	411·25
„ „	58·49	−104·32 h − 78·81 k	+183·41 h + 149·16 k	+287·73 h + 227·97 k	410·65
„ „	58·53	− 82·40 h − 99·63 k	+198·77 h + 134·22 k	+281·17 h + 233·85 k	410·12
Oct. 1	58·33	−139·00 h − 39.20 k	+158·90 h + 177·89 k	+297·90 h + 217·09 k	410·05
„ „	58·44	−157·00 h − 23·43 k	+131·21 h + 205·88 k	+288·21 h + 229·31 k	412·10
„ „	58·54	−117·42 h − 62·88 k	+150·68 h + 184·98 k	+268·10 h + 247·86 k	410·91
„ 2	58·50	−103·96 h − 76·93 k	+192·40 h + 142·78 k	+296·36 h + 219·71 k	410·92
„ 20	63·16	−154·18 h − 100·54 k	+ 56·62 h + 206·02 k	+210·80 h + 306·56 k	412·21
„ „	63·31	−127·14 h − 130·61 k	+129·29 h + 131·19 k	+256·43 h + 261·80 k	412·77
„ „	63·37	−131·42 h − 125·19 k	+127·27 h + 133·28 k	+258·69 h + 258·47 k	411·90
„ 21	62·39	−137·21 h − 103·56 k	+119·37 h + 155·21 k	+256·58 h + 258·77 k	410·47
„ „	62·39	−129·01 h − 112·11 k	+129·89 h + 146·99 k	+258·90 h + 259·10 k	412·56
„ „	62·49	−128·02 h − 113·62 k	+127·04 h + 149·47 k	+255·06 h + 263·09 k	412·71
„ 22	62·03	−131·21 h − 105·78 k	+132·78 h + 146·84 k	+263·99 h + 252·62 k	411·45
„ „	62·12	−118·31 h − 120·43 k	+141·78 h + 135·89 k	+260·09 h + 256·32 k	411·31
„ 23	61·61	−129·52 h − 105·48 k	+134·04 h + 148·97 k	+263·56 h + 254·45 k	412·56
„ „	61·63	−123·29 h − 110·63 k	+136·48 h + 146·02 k	+259·77 h + 256·65 k	411·31
1864:					
Jan. 5	34·48	−133·12 h − 113·28 k	+126·88 h + 134·06 k	+260·00 h + 247·34 k	404·06
„ „	34·61	−141·28 h − 107·49 k	+125·46 h + 132·19 k	+266·74 h + 239·68 k	403·31
„ 6	32·76	−113·34 h − 120·15 k	+149·32 h + 125·12 k	+262·66 h + 245·27 k	404·53
„ „	32·79	−117·11 h − 115·60 k	+129·81 h + 143·21 k	+246·92 h + 258·81 k	402·82
„ „	32·93	−119·56 h − 117·41 k	+136·98 h + 133·09 k	+256·54 h + 250·50 k	403·83
„ 7	31·29	−112·90 h − 107·28 k	+117·46 h + 165·69 k	+230·36 h + 272·97 k	400·79
„ „	31·09	−110·89 h − 109·48 k	+138·94 h + 147·71 k	+249·83 h + 257·19 k	403·84
„ „	31·41	−116·02 h − 110 20 k	+136·66 h + 143·09 k	+252·68 h + 253·29 k	403·00

RESULTS OF COMPARISON &c.—*continued.*

Date.		Mean Temp.	Prussian Toise.	Ordnance Toise.	Difference in Micrometer Divisions.	$T_{10} - T_0 + \sigma$.
1864:		°				
Jan	8	30·79	$-108{\cdot}43\,h - 114{\cdot}41\,k$	$+135{\cdot}94\,h + 147{\cdot}09\,k$	$+244{\cdot}37\,h + 261{\cdot}50\,k$	402·94
„	„	30·87	$-110{\cdot}06\,h - 113{\cdot}36\,k$	$+134{\cdot}62\,h + 146{\cdot}69\,k$	$+244{\cdot}68\,h + 260{\cdot}05\,k$	402·03
„	„	31·43	$-113{\cdot}88\,h - 119{\cdot}39\,k$	$+138{\cdot}73\,h + 135{\cdot}05\,k$	$+252{\cdot}61\,h + 254{\cdot}44\,k$	403·86
„	9	32·22	$-114{\cdot}98\,h - 130{\cdot}80\,k$	$+129{\cdot}12\,h + 132{\cdot}62\,k$	$+244{\cdot}10\,h + 263{\cdot}42\,k$	404·26
„	„	32·65	$-120{\cdot}91\,h - 128{\cdot}86\,k$	$+130{\cdot}09\,h + 127{\cdot}82\,k$	$+251{\cdot}00\,h + 256{\cdot}68\,k$	404·36
„	„	33·03	$-112{\cdot}39\,h - 140{\cdot}26\,k$	$+129{\cdot}32\,h + 124{\cdot}93\,k$	$+241{\cdot}71\,h + 265{\cdot}19\,k$	403·76
„	„	33·35	$-126{\cdot}80\,h - 131{\cdot}32\,k$	$+123{\cdot}58\,h + 127{\cdot}97\,k$	$+250{\cdot}38\,h + 259{\cdot}29\,k$	405·96
May	12	62·73	$-134{\cdot}44\,h - 131{\cdot}54\,k$	$+128{\cdot}46\,h + 122{\cdot}10\,k$	$+262{\cdot}90\,h + 253{\cdot}64\,k$	411·40
„	„	62·86	$-134{\cdot}88\,h - 132{\cdot}98\,k$	$+124{\cdot}81\,h + 123{\cdot}23\,k$	$+259{\cdot}69\,h + 256{\cdot}21\,k$	410·90
„	„	63·00	$-135{\cdot}87\,h - 131{\cdot}08\,k$	$+121{\cdot}46\,h + 127{\cdot}36\,k$	$+257{\cdot}33\,h + 258{\cdot}44\,k$	410·80
„	13	63·45	$-134{\cdot}21\,h - 138{\cdot}71\,k$	$+121{\cdot}19\,h + 123{\cdot}38\,k$	$+255{\cdot}40\,h + 262{\cdot}09\,k$	412·18
„	„	63·52	$-137{\cdot}73\,h - 137{\cdot}77\,k$	$+116{\cdot}67\,h + 125{\cdot}40\,k$	$+254{\cdot}40\,h + 263{\cdot}17\,k$	412·24
„	„	63·57	$-138{\cdot}16\,h - 133{\cdot}99\,k$	$+121{\cdot}74\,h + 122{\cdot}76\,k$	$+259{\cdot}90\,h + 256{\cdot}75\,k$	411·49
„	„	63·66	$-135{\cdot}01\,h - 137{\cdot}06\,k$	$+122{\cdot}59\,h + 122{\cdot}00\,k$	$+257{\cdot}60\,h + 259{\cdot}06\,k$	411·51
„	14	63·46	$-135{\cdot}01\,h - 134{\cdot}34\,k$	$+126{\cdot}02\,h + 122{\cdot}27\,k$	$+261{\cdot}03\,h + 256{\cdot}61\,k$	412·28
„	„	63·51	$-134{\cdot}52\,h - 136{\cdot}87\,k$	$+120{\cdot}76\,h + 127{\cdot}22\,k$	$+255{\cdot}28\,h + 264{\cdot}09\,k$	413·68
„	„	63·58	$-133{\cdot}20\,h - 135{\cdot}58\,k$	$+123{\cdot}79\,h + 126{\cdot}93\,k$	$+256{\cdot}99\,h + 262{\cdot}51\,k$	413·78

Temperature corrected for errors of thermometers.

Each line of this Table gives us a result of the form,

$$T_{10} + \sigma - T_0 = a$$

where a varies with the temperature. Let y be the excess of the expansion (for 1° Fahrenheit) of the Prussian bar above the Ordnance, and put

$$T_{10} + \sigma - T_0 = 400 + x + fy$$

where f is the excess of the temperature of the bars at the time of observation above 62°, then every comparison gives an equation of the form

$$x + fy - a = 0 \qquad (1)$$

Treating the 65 equations thus formed by the method of least squares we get for the determination of x and y the equations

$$65\,x - 462{\cdot}07\,y - 625{\cdot}50 = 0 \qquad (2)$$
$$-462{\cdot}07\,x + 13333{\cdot}54\,y + 1756{\cdot}59 = 0$$

If we write A and B for the absolute terms of these equations, we get

$$x + {\cdot}0204135\,A + {\cdot}0007074\,B = 0 \qquad (3)$$
$$y + {\cdot}0007074\,A + {\cdot}0000995\,B = 0$$

Restoring the values of A and B we find

$$x = 11{\cdot}526 \qquad (4)$$
$$y = 0{\cdot}2677$$

and for the reciprocals of the weights of x, y, and $x + fy$

$$x \;.\;.\;.\;.\;.\; {\cdot}02041 \qquad (5)$$
$$y \;.\;.\;.\;.\;.\; {\cdot}00010$$
$$x + fy \;.\;.\;.\;.\;.\; {\cdot}02041 + {\cdot}00141\,f + {\cdot}00010\,f^2$$

Substituting in the sixty-five equations the above values of x and y we get their errors, or the errors of the comparisons, as shown in the following Table :—

Date.	Error.	Date.	Error.	Date.	Error.
1863 :		1863:		1864:	
September 11	− 2·28	September 29	+ 0·09	January 7	− 2·52
" "	− 1·36	" "	+ 0·72	" "	+ 0·58
" "	− 2·47	" "	+ 1·50	" "	− 0·34
" "	− 2·39	" 30	+ 0·68	" 8	− 0·24
" 12	− 0·75	" "	+ 0·06	" "	− 1·17
" "	− 1·22	" "	− 0·48	" "	+ 0·51
" "	− 1·88	October 1	− 0·50	" 9	+ 0·70
" 14	− 2·01	" "	+ 1·52	" "	+ 0·69
" "	− 1·36	" "	+ 0·31	" "	− 0·01
" "	− 1·63	" 2	+ 0·33	" "	+ 2·10
September 18	+ 1·07	October 20	+ 0·37	May 12	− 0·33
" 19	− 0·09	" "	+ 0·89	" "	− 0·86
" "	+ 1·78	" "	+ 0·00	" "	− 1·00
" 21	+ 1·05	" 21	− 1·16	" 13	+ 0·26
" "	+ 0·77	" "	+ 0·93	" "	+ 0·30
" "	+ 0·97	" "	+ 1·05	" "	− 0·46
" 22	− 0·17	" 22	− 0·09	" "	− 0·46
" "	+ 0·95	" "	− 0·25	" 14	+ 0·36
" "	+ 1·70	" 23	+ 1·13	" "	+ 1·75
" 23	+ 1·36	" "	− 0·12	" "	+ 1·83
		1864:			
		January 5	− 0·10		
		" "	− 0·89		
		" 6	+ 0·83		
		" "	− 0·89		
		" "	+ 0·08		

The sum of the squares of these errors is 82·589. Hence the probable error of a single comparison is

$$\pm 0\cdot674\sqrt{\frac{82\cdot859}{65-2}} = \pm 0\cdot772 \qquad (6)$$

Now, if we count the number of errors which are greater, and the number that are less than the computed probable error we find 32 errors less than ·772 and 33 errors greater than ·772; and so far all is in accordance with the supposition of perfectly accidental errors. But on examination it will be seen that the errors in the first series are all of one sign as though affected by some constant error. The only cause that suggests itself is that the bar was supported on rollers which were true cylinders. On the other hand, the makers, Messrs. Troughton and Simms, were very careful to adjust these rollers into parallelism, and it is, therefore, scarcely safe to conclude that this is the cause. In the second series there is a preponderance of positive errors suggesting a possible constant error here also. If by the value of y which has been obtained we correct all the comparisons to 62°, the different series give by these means the following results :—

Series.	$T_{10} + \sigma - T_0$ at 62°.	Error.
1	409·80	− 1·73
2	412·47	+ 0·94
3	411·95	+ 0·42
4	411·80	+ 0·27
5	411·49	− 0·04
6	411·67	+ 0·14

Taking the mean of the determinations in the first and second series, we obtain as the value of $\mathbf{T}_{10} + \sigma - \mathbf{T}_0$ at 62°, 411·14, which does not differ materially from the results in the other series. The errors, therefore, in the second series nearly balance those in the first, and for this reason we shall not reject the first series, although there is a strong temptation to do so.

Taking $\pm$ 0·772 as the probable error of single comparison or equation, the probable error of x

$$= \pm 0{\cdot}772 \sqrt{\cdot 02041} = \pm 0{\cdot}110 \tag{7}$$

and the probable error of y

$$= \pm 0{\cdot}772 \sqrt{\cdot 00010} = \pm 0{\cdot}0077 \tag{8}$$

So that our final results are—

$$\mathbf{T}_{10} - \mathbf{T}_0 = 411{\cdot}53 \pm 0{\cdot}11 - \sigma \tag{9}$$

at the temperature of 62° Fahrenheit. And the excess of the expansion of $\mathbf{T}_{10}$ above that of $\mathbf{T}_0$ for 1° of temperature

$$= 0{\cdot}2677 \pm 0{\cdot}0077 \tag{10}$$

If the first series were rejected, we should have obtained $\mathbf{T}_{10} - \mathbf{T}_0 = 411{\cdot}98 \pm {\cdot}095 - \sigma$ at 62°; and the difference of expansion = 0·2842 $\pm$ ·0061.

3.

In the description of the contact apparatus it has been shown, that if, when the parallel or longitudinal and transverse lines were drawn, the centres of the semicylinders were not in a line parallel to the direction of motion of the needles, but were moveable in two distinct parallel lines as represented in figure 2, page 12; then the distance of the transverse lines from one another will be increased when the semicylinders are brought exactly opposite to one another by the amount

$$\frac{\varrho}{4\, i^2} (\delta_{,} - \delta')^2$$

where $\delta_{,}$ and δ' are the distances apart of the transverse lines, first, when (by the motion of the transverse slow motion screw) the lines a and b' are brought into one straight line, and, second, when the lines a' and b are in one straight line. If then δ be the distance apart of the transverse lines when the longitudinal lines are normally adjusted as in figure 2, the maximum distance or the quantity σ $(= \lambda + \lambda')$ is

$$\sigma = \delta + \frac{\varrho}{4\, i^2} (\delta_{,} - \delta')^2 \tag{11}$$

We shall first consider the measurement of the quantity δ. The approximate value of this space is $\frac{1}{50}$ of an inch from which it differs but very little. The easiest way of measuring this would be simply to run it over with the micrometer screw of either or both microscopes, but the largeness of the space requires that the value of a division be very accurately known: δ being over 700 divisions. By the formula (23) page 63, it would appear that the probable error of a *single* measurement of this space with the micrometer is

$$\text{for H} \ldots\ldots \pm \sqrt{\cdot 187 \left(\tfrac{7}{10}\right)^2 + \cdot 20} = \pm 0{\cdot}54$$
$$\text{for K} \ldots\ldots \pm \sqrt{\cdot 347 \left(\tfrac{7}{10}\right)^2 + \cdot 20} = \pm 0{\cdot}61$$

From this we might conclude that not much error need be feared from the actual measurement of the space by the micrometers; yet it was thought better to compare the space δ with some of the fiftieths of an inch in the foot **O F**. The five equal subdivisions of the tenth of an inch [6·7] on **O F** were accordingly chosen; and on four different occasions, September 15th, 24th, 26th, and October 2d, the space δ was compared with each of these five spaces. In Table VIII., page 59, we have the errors of the subdivisions expressed in the micrometer divisions. The errors of the four lines subdividing [6·7] into fiftieths are

$$-\tfrac{1}{2}(0{\cdot}99\,h + 0{\cdot}61\,k) = -0{\cdot}64$$
$$-\tfrac{1}{2}(2{\cdot}31\,h + 1{\cdot}70\,k) = -1{\cdot}60$$
$$-\tfrac{1}{2}(3{\cdot}56\,h + 2{\cdot}97\,k) = -2{\cdot}60$$
$$-\tfrac{1}{2}(3{\cdot}14\,h + 2{\cdot}55\,k) = -2{\cdot}27$$

Consequently the errors of the five spaces *as fifths of* [6·7] are

$$\begin{array}{ll} \text{1st} \ldots\ldots & -0{\cdot}64 \\ \text{2d} \ldots\ldots & -0{\cdot}96 \\ \text{3d} \ldots\ldots & -1{\cdot}00 \\ \text{4th} \ldots\ldots & +0{\cdot}33 \\ \text{5th} \ldots\ldots & +2{\cdot}27 \end{array} \qquad (12)$$

The space δ was compared with each of these fiftieths of an inch in precisely the same manner as the tenths of inches on **O F** were compared with $\frac{2}{10}$ of an inch and $\frac{3}{10}$ of an inch on the small silver scale, see page 52. The only difference is, that the contact apparatus was on the *right* of the foot **O F**. See fig. 5, Plate VI.

Each comparison was conducted exactly as in the comparison of two bars, viz., in the following order:—

{ Left lines,	3 readings of H,	and	3 readings of K		
{ Right lines,	"	"	"	"	
{ Right lines,	"	"	"	"	
{ Left lines,	"	"	"	"	
{ Left lines,	"	"	"	"	
{ Right lines,	"	"	"	"	

In all, 36 micrometer readings, forming *one comparison*.

The following Table contains the results of the twenty comparisons. The sixth column contains the correction from equation (12), to be applied for the errors of the spaces as fifths of [6·7]. The seventh column contains the corrected value of the excess of δ over one-fiftieth of an inch.

Space.	Left Lines.	Right Lines.	L. − R. in Micr. Divisions.	Diff. of Length.	Corrections.	Corrected Diff. in Length.
First $\frac{2}{100}$	$+ 0{\cdot}70h - 15{\cdot}17k$	$-12{\cdot}60h - 14{\cdot}17k$	$+13{\cdot}30h - 1{\cdot}00k$	+ 9·77	−0·64	9·13
	$+ 9{\cdot}20h + 23{\cdot}03k$	$+ 3{\cdot}80h + 14{\cdot}73k$	$+ 5{\cdot}40h + 8{\cdot}30k$	+ 10·91	,,	10·27
	$+20{\cdot}63h + 8{\cdot}63k$	$+ 4{\cdot}70h + 10{\cdot}50k$	$+15{\cdot}93h - 1{\cdot}87k$	+ 11·17	,,	10·53
	$+ 1{\cdot}67h + 55{\cdot}10k$	$+ 0{\cdot}17h + 43{\cdot}70k$	$+ 1{\cdot}50h + 11{\cdot}40k$	+ 10·29	,,	9·65
Second $\frac{2}{100}$	$-12{\cdot}50h + 2{\cdot}40k$	$-16{\cdot}24h - 8{\cdot}00k$	$+ 3{\cdot}74h + 10{\cdot}40k$	+ 11·27	−0·96	10·31
	$+ 0{\cdot}87h + 31{\cdot}13k$	$- 1{\cdot}57h + 18{\cdot}57k$	$+ 2{\cdot}44h + 12{\cdot}56k$	+ 11·96	,,	11·00
	$+16{\cdot}80h + 10{\cdot}13k$	$+ 1{\cdot}87h + 10{\cdot}37k$	$+14{\cdot}93h - 0{\cdot}24k$	+ 11·68	,,	10·72
	$+36{\cdot}93h + 23{\cdot}47k$	$+23{\cdot}83h + 22{\cdot}30k$	$+13{\cdot}10h + 1{\cdot}17k$	+ 11·34	,,	10·38
Third $\frac{2}{100}$	$+ 6{\cdot}87h + 5{\cdot}37k$	$- 6{\cdot}27h + 3{\cdot}07k$	$+13{\cdot}14h + 2{\cdot}30k$	+ 11·29	−1·00	11·29
	$+ 0{\cdot}57h + 32{\cdot}43k$	$+ 6{\cdot}13h + 11{\cdot}83k$	$- 5{\cdot}56h + 20{\cdot}60k$	+ 12·02	,,	11·02
	$+24{\cdot}20h + 8{\cdot}17k$	$+ 8{\cdot}63h + 8{\cdot}90k$	$+15{\cdot}57h - 0{\cdot}73k$	+ 11·80	,,	10·80
	$+18{\cdot}13h + 28{\cdot}63k$	$+ 3{\cdot}60h + 29{\cdot}00k$	$+14{\cdot}53h - 0{\cdot}37k$	+ 11·25	,,	10·25
Fourth $\frac{2}{100}$	$+ 8{\cdot}97h + 1{\cdot}87k$	$+ 0{\cdot}03h - 0{\cdot}23k$	$+ 8{\cdot}94h + 2{\cdot}10k$	+ 8·79	+0·33	9·12
	$+16{\cdot}77h + 8{\cdot}23k$	$+ 3{\cdot}37h + 8{\cdot}83k$	$+13{\cdot}40h - 0{\cdot}60k$	+ 10·17	,,	10·50
	$+21{\cdot}57h + 8{\cdot}17k$	$+ 8{\cdot}60h + 7{\cdot}73k$	$+12{\cdot}97h + 0{\cdot}44k$	+ 10·66	,,	10·99
	$+17{\cdot}07h + 32{\cdot}90k$	$+ 5{\cdot}13h + 31{\cdot}83k$	$+11{\cdot}94h + 1{\cdot}07k$	+ 10·34	,,	10·67
Fifth $\frac{2}{100}$	$+13{\cdot}03h - 3{\cdot}27k$	$+ 1{\cdot}87h - 1{\cdot}20k$	$+11{\cdot}16h - 2{\cdot}07k$	+ 7·22	+2·27	9·49
	$+11{\cdot}47h + 14{\cdot}60k$	$+ 0{\cdot}03h + 14{\cdot}33k$	$+11{\cdot}44h + 0{\cdot}27k$	+ 9·30	,,	11·57
	$+30{\cdot}50h + 2{\cdot}50k$	$+18{\cdot}50h + 5{\cdot}17k$	$+12{\cdot}00h - 2{\cdot}67k$	+ 7·41	,,	9·68
	$+20{\cdot}97h + 32{\cdot}57k$	$+ 9{\cdot}93h + 34{\cdot}50k$	$+11{\cdot}04h - 1{\cdot}93k$	+ 7·24	,,	9·51

The mean of the quantities in the last column is 10·34, and the sum of the squares of the errors is 9·55. Hence the probable error of a single determination is—

$$\pm \cdot 674 \sqrt{\frac{9\cdot 55}{20 - 1}} = \pm 0\cdot 478 \qquad (13)$$

and the probable error of the final determination

$$\pm \frac{\cdot 478}{\sqrt{20}} = \pm 0\cdot 107 \qquad (14)$$

We have then

$$\delta = \tfrac{1}{5}\,[6\cdot 7] + 10\cdot 34 \qquad (15)$$

Here δ is composed of two parts, and has consequently two partial probable errors. The probable error of the second part results from the comparisons in Table H., and we have found it to be $\pm \cdot 107$. The probable error of the first part depends on the probable error of the determination of the value of $[6\cdot 7]$ on **O F** or rather is affected by one-fifth part of that error. The errors of the determinations of the subdivisions of $[6\cdot 7]$ do not enter. Now, by equations (3), page 49, $[6\cdot 7]$ is expressed thus:—

$$[6\cdot 7] = \frac{\mathsf{F}}{120} - x_6 + x_7 + \frac{x_b}{10} + \frac{x_a}{60} = \frac{\mathsf{F}}{120} - 1\cdot 42$$

From equations (12), page 56, the reciprocal of the weight of $- x_6 + x_7$ is

$$\frac{28}{40} - \frac{12}{40} + \frac{27}{40} = \frac{43}{40}$$

consequently by equation (32), page 65, the probable error of the determination of $-x_6 + x_7$ is

$$\pm \frac{\cdot 386}{5} \sqrt{\frac{43}{40}} = \pm \cdot 080$$

The probable errors of x_b and x_g were shown to be $\pm \cdot 036$ and $\pm \cdot 037$; hence finally the probable error of the determination of [6·7] is

$$\pm \sqrt{(\cdot 080)^2 + (\cdot 004)^2 + (\cdot 001)^2} = \pm \cdot 080$$

and

$$\tfrac{1}{5}[6\cdot 7] = \frac{F}{600} - 0\cdot 28 \pm \cdot 016 \qquad (16)$$

Thus, the value of δ becomes

$$\delta = \frac{F}{600} + 10\cdot 06 \pm \sqrt{(\cdot 107)^2 + (\cdot 016)^2}$$

Now, expressed in millionths of a yard $F = 333333\cdot 33$ and $\frac{1}{600} F = 555\cdot 56$,

$$\therefore \delta = 565\cdot 62 \pm 0\cdot 108 \qquad (17)$$

It remains to compute the small correction to this quantity δ which is exhibited in equation (11). In order to obtain $\delta_, - \delta'$ both $\delta_,$ and δ' were measured by runs of the micrometer H. Each quantity was measured ten times and they were measured alternately, first, $\delta_,$ then δ' and so on. Thus, each measure was entirely independent as far as contact and adjustment of the longitudinal lines into *one*, are concerned. But the focal adjustment remained constant through the series, purposely, this error disappearing in $\delta_, - \delta'$. The following Table contains the result of these measures; each number is the difference between the mean of two readings of the left line and the mean of two readings of the right line.

Divisions of Micrometer H.	
$\delta_,$	δ'
d	d
695·3	702·8
695·9	701·2
695·6	701·8
694·3	701·4
694·5	702·4
694·6	702·6
694·8	702·7
694·5	702·3
694·3	702·3
694·2	702·2

Therefore

$$\begin{aligned} \delta_, &= 694\overset{d}{\cdot}80 \pm 0\cdot 127 \\ \delta' &= 702\cdot 17 \pm 0\cdot 115 \\ \delta_, - \delta' &= \quad 7\cdot 37 \pm 0\cdot 171 \end{aligned}$$

or, expressed in millionths of a yard

$$\delta - \delta' = 5\cdot 86 \pm 0\cdot 136 \qquad (18)$$

The distance apart of the parallel longitudinal lines ($= i$) is 775 divisions of H or 616·0 in our unit; also the radius ϱ of the cylinders is 0·365 inch, which referred to the unit

$$= \frac{365}{1000} \cdot \frac{1000000}{36} = 10139$$

$$\therefore \frac{\varrho}{4\, i^2} (\delta_{,} - \delta')^2 = 2535 \left(\frac{5 \cdot 86 \pm \cdot 136}{616}\right)^2$$

$$\frac{\varrho}{4\, i^2} (\delta_{,} - \delta')^2 = 0 \cdot 23 \pm 0 \cdot 011 \qquad (19)$$

The correction is, therefore, a very small quantity, and its probable error we may neglect. Hence, finally, from (11), (17), (19),

$$\sigma = 565 \cdot 85 \pm 0 \cdot 108 \qquad (20)$$

4.

If in equation (9) we substitute the value of σ given in equation (20) we get

$$\mathbf{T}_{10} - \mathbf{T}_0 = -154 \cdot 32 \pm \sqrt{(\cdot 110)^2 + (\cdot 108)^2}$$

$$\therefore \mathbf{T}_{10} = \mathbf{T}_0 - 154 \cdot 32 \pm 0 \cdot 154$$

For the value of $\mathbf{T}_0$ we refer to page 144, where equation (16) gives

$$\mathbf{T}_0 = (2 \cdot 13167512 \pm \cdot 000000214)\, \mathbf{Y}_{55}$$

from which there follows

$$\mathbf{T}_{10} = 2 \cdot 13152080\, \mathbf{Y}_{55}$$

5.

With respect to the probable error of the result at which we have now arrived, it will be necessary in order to obtain it with precision, to review the whole of the operations which have led to the number 2·13152080, and to show the manner in which it is connected with each of these operations, and the extent to which it is dependent on them severally.

Retaining the notation of equations (1), (2), (3), pages 48, 49, we have by equation (42), page 70,

$$[\tau . f] = \frac{474}{1200}\, \mathbf{F} - \frac{4}{10}\, x_2 - x_\tau - \frac{6}{10}\, x_3 - \frac{26}{100}\, x_b + x_f + \frac{474}{600}\, x_g$$

Thus the determination of the length of $[\tau . f]$ as a portion of **O F** depends on four entirely independent operations.

1. The bisection of **O F**,
2. The subdivision of one-half of **O F** into six equal parts,
3. The subdivision of one of these sixths or inches into ten,
4. The subdivision of one of these last parts into ten.

The corresponding parts of the above expression for $[\tau . f]$ are as follows :—

$$1 \ldots\ldots \frac{474}{630} x_g$$

$$2 \ldots\ldots -\frac{26}{100} x_b + x_f$$

$$3 \ldots\ldots -\frac{4}{10} x_2 - \frac{6}{10} x_3$$

$$4 \ldots\ldots - x_\tau$$

Let λ_1 be the excess of **F** over $\frac{1}{3}$ Y_{65} at 62°, ϵ_1 the excess of the expansion of **F** over $\frac{1}{3}$ Y_{65} for each degree Fahrenheit, then

$$F = \frac{1}{3} Y_{65} + \lambda_1 + \epsilon_1 (t - 62)$$

Let λ_2 represent the excess of the length of the two adjacent yards on **O T** above 2 Y_{65} at 62°, and let ϵ_2 represent the excess of the expansion of **Y** above one yard of **O T** then the length of the two yards of **O T**

$$= 2 Y_{65} + \lambda_2 - 2 \epsilon_2 (t - 62)$$

Let λ_3 be the excess of length of the small space of 4·74 inches on **O T** above $[\tau \cdot f]$ on **O F** at the temperature 62° + b, then the excess of the space on **O T** above the space on **O F** at 62° is

$$\lambda_3 + b \frac{474}{1200} (\epsilon_1 + \frac{1}{3} \epsilon_2)$$

Hence the length of **O T** at 62° is

$$2 Y_{65} + [\tau \cdot f] + \lambda_2 + \lambda_3 + b \frac{474}{1200} (\epsilon_1 + \frac{1}{3} \epsilon_2)$$

Now at 62°

$$[\tau . f] = \frac{474}{3600} Y_{65} + \frac{474}{1200} \lambda_1 - \frac{4}{10} x_2 - x_\tau - \frac{6}{10} x_3 - \frac{26}{100} x_b + x_f + \frac{474}{600} x_g$$

And the consequent length of the Ordnance Toise is, at 62°,

$$\frac{7674}{3600} Y_{65} + \frac{474}{1200} \lambda_1 + \lambda_2 + \lambda_3 + \frac{474}{1200} b (\epsilon_1 + \frac{1}{3} \epsilon_2) - \frac{4}{10} x_2 - x_\tau - \frac{6}{10} x_3 - \frac{26}{100} x_b + x_f + \frac{474}{600} x_g$$

Thus the length of **O T** depends on the four operations enumerated above, together with the following :—

5. The comparisons of **O F** with Y_{65}
6. The comparisons of the two yards of **O T** with Y_{65}
7. The comparison of the small space on **O T** with the corresponding space on **O F**

Again, the distance of the two lines of the contact apparatus when brought into conjunction has been found to exceed the fifth part of [6·7] by a quantity, say λ_4

$$\sigma = \frac{1}{5} [6 \cdot 7] + \lambda_4$$

$$= \frac{F}{600} - \frac{1}{5} x_6 + \frac{1}{5} x_7 + \frac{x_b}{50} + \frac{x_g}{300} + \lambda_4$$

$$= \frac{Y_{65}}{1800} + \frac{\lambda_1}{600} + \lambda_4 - \frac{1}{5} x_6 + \frac{1}{5} x_7 + \frac{x_b}{50} + \frac{x_g}{300}$$

Finally, let λ_5 be the excess of the length of the Prussian Toise $+ \sigma$, above the Ordnance Toise at 62°, then

$$\mathbf{T}_{10} - \mathbf{T}_0 = \lambda_5 - \sigma$$
$$= - \frac{\mathbf{Y}_{55}}{1800} - \frac{\lambda_1}{600} - \lambda_4 + \lambda_5 + \tfrac{1}{5} x_6 - \tfrac{1}{5} x_7 - \frac{x_b}{50} - \frac{x_g}{300}$$

where two further operations are involved; viz.,

8. The determination of the value of the space on the contact apparatus.
9. The comparison of the Ordnance and Prussian Toises.

We have then the following expression for the length of the Prussian Toise at 62°

$$\left.\begin{array}{l} \frac{7672}{3600} \mathbf{Y}_{55} + \frac{472}{1200} \lambda_1 + \lambda_2 + \lambda_3 - \lambda_4 + \lambda_5 + \frac{474}{1200}\, b \left(\epsilon_1 + \frac{\epsilon_2}{3} \right) \\ \qquad - \frac{4}{10} x_2 - x_\tau - \frac{6}{10} x_3 + \frac{1}{5} x_6 - \frac{1}{5} x_7 - \frac{28}{100} x_b + x_f + \frac{472}{600} x_g \end{array}\right\}$$

The parts of this formula which appertain to the several operations in the order enumerated above are

1. $\frac{472}{600} x_g$
2. $- \frac{28}{100} x_b + x_f$
3. $\frac{1}{5} (- 2 x_2 - 3 x_3 + x_6 - x_7)$
4. $- x_\tau$
5. $\frac{472}{1200} \lambda_1 + \frac{474}{1200} b \epsilon_1$
6. $\lambda_2 + \frac{474}{3600} b \epsilon_2$
7. λ_3
8. $- \lambda_4$
9. λ_5

Now by equations 6, page 51, the reciprocal of the weight of $\alpha x_b + x_f$ is

$$\frac{11 \alpha^2 + 2 \alpha + 11}{18}$$

and if $\alpha = - \frac{28}{100}$ this becomes $\frac{113024}{180000}$, and consequently the probable error of the determination

$$\pm \frac{\cdot 233}{5} \sqrt{\frac{113}{180}} = \pm \cdot 037$$

Again, from equations (12), page 56, the reciprocal of the weight of the determination of $\alpha x_2 + \beta x_3 + \gamma x_6 + \delta x_7$ is

$$\left.\begin{array}{l} \frac{27}{2} \alpha^2 + 3 \alpha \beta + 6 \alpha \gamma + 7 \alpha \delta \\ \qquad + \frac{27}{2} \beta^2 + 14 \beta \gamma + 3 \beta \delta \\ \qquad\qquad + 14 \gamma^2 + 6 \gamma \delta \\ \qquad\qquad\qquad + \frac{27}{2} \delta^2 \end{array}\right\} \div 20$$

And making $\alpha = -2$; $\beta = -3$; $\gamma = 1$; $\delta = -1$, the reciprocal of the weight of the determination of $-2x_2 - 3x_3 + x_6 - x_7$ becomes

$$\frac{184}{20}$$

and the probable error of the determination of $-2x_2 - 3x_3 + x_6 - x_7$

$$= \pm \frac{\cdot 386}{5}\sqrt{\frac{92}{10}} = \pm 0\cdot 234$$

and the probable error of one-fifth of this is

$$\pm 0\cdot 047$$

The reciprocal of the weight of $\lambda_1 + \theta\,\varepsilon_1$ by equation (3), page 76, is

$$\cdot 10723 + \cdot 01150\,\theta + \cdot 000447\,\theta^2$$

now $b = \cdot 53$, and if $\theta = \frac{474}{472}\,b$ this becomes $= \cdot 11348$, and the probable error of $\lambda_1 + \frac{474}{472}\,b\,\varepsilon_1$ is

$$\pm 0\cdot 330\sqrt{\cdot 11348} = \pm 0\cdot 111$$

and the probable error of

$$\frac{472}{1200}\lambda_1 + \frac{474}{1200}\,b\,\varepsilon_1$$

becomes

$$\pm \frac{472}{1200} \times 0\cdot 111 = \pm 0\cdot 044$$

Again by equation (8), page 100, it appears that the reciprocal of weight of the determination of $x_, + x' + \theta\,y$ is

$$\cdot 12666 + \cdot 00663\,\theta + \cdot 000183\,\theta^2$$

In the formula we are now considering, λ_2 corresponds to $x_, + x'$, and ε_2 to $-y$, we must therefore in order to obtain the reciprocal of the weight of the determination of

$$\lambda_2 + \frac{474}{3600}\,b\,\varepsilon_2$$

make

$$\theta = -\frac{474}{3600}\,b = -\frac{474 \times 53}{360000}$$

which gives for the reciprocal of the weight

$$\cdot 12620$$

and therefore the probable error of $\lambda_2 + \frac{474}{3600}\,b\,\varepsilon_2$ is by equation (10), page 101,

$$\pm \cdot 518\sqrt{\cdot 12620} = \pm 0\cdot 184$$

The following Table shows the amount of probable error due to the nine operations.

Nature of Operation.	Probable Error.
1. The bisection of the foot **O F** - - -	± ·029
2. The subdivision of one-half of **O F** into six equal parts -	± ·037
3. The subdivision of one of these parts into ten - -	± ·047
4. The subdivision of one of these last parts again into ten -	± ·098
5. The comparison of **O F** with $\mathbf{Y}_{55}$ - - - -	± ·044
6. The comparison of **O T** with $\mathbf{Y}_{55}$ - - -	± ·184
7. The comparison of **O T** with **O F** - - - - -	± ·064
8. The measurement of the space on the contact apparatus -	± ·107
9. The comparison of **O T** with the Prussian Toise - -	± ·110

$$(\cdot 029)^2+(\cdot 037)^2+(\cdot 047)^2+(\cdot 098)^2+(\cdot 044)^2+(\cdot 184)^2+(\cdot 064)^2+(\cdot 107)^2+(\cdot 110)^2$$
$$=(\cdot 278)^2$$

Hence the length of the Prussian Toise in terms of the Copy of the Standard Yard No. 55 both being at 62° Fahrenheit is

$$\mathbf{T}_{10}=(2\cdot 13152080\pm\cdot 00000028)$$

XII.

DETERMINATION OF THE LENGTH OF THE BELGIAN TOISE.

The Standard Toise of the "Dépôt de la Guerre" of Belgium is in all respects similar to the Prussian Toise which has been already described. They were made by the same maker nearly at the same time, and are according to the determination of General Baeyer almost exactly of the same length. The only difference is that one is marked No. 10 and the other, the Belgian, is No. 11.

The Toise was lent to this Department in 1862 by General Nerenburger and was brought to Southampton by Captain Libois in the month of September that its length might be obtained in terms of the Standard Yard. Being the first experiment in comparisons between a measure *à traits* and a measure *à bouts* much difficulty was found in getting satisfactory results. The computed probable error of the resulting length was $2 \cdot 13$ (millionths of a yard). This is rather a large quantity for a computed probable error, and moreover the manifest imperfections of the means then available left much to be desired.

Accordingly, in September 1864, the Toise was again, at the request of Colonel Sir Henry James, R.E., lent by the Minister of War at Brussels, General Simon, for comparison with the Ordnance Toise in the same manner and with the same apparatus as the Prussian Toise had been compared. The Toise No. 11 arrived at Southampton on the morning of 29th September, and was immediately placed on the apparatus, and the comparisons commenced next day.

1.

In order to guard against any possible change of length in the Ordnance Toise, **O T**, which might have taken place since October 1863, it was considered advisable to examine each of its three spaces $[\alpha \cdot \beta]$, $[\beta \cdot \gamma]$, $[\gamma \cdot \delta]$ previously to the comparison with the Belgian Toise. A series of ten comparisons of $[\alpha \cdot \beta]$ with the copy of the Standard Yard No. 55 was made accordingly on September 19th, 20th, 22d, 23d: ten comparisons of $[\beta \, \gamma]$ were also made on the 21st, 22d, and 24th, where by a comparison is meant the mean result of one visit.

On 26th September ten *single comparisons* of $[\gamma \cdot \delta]$ with the space $[\tau \cdot f]$ on the bar **O T** were made. Here by *single comparison* is meant three micrometer readings with each microscope of the one bar, and the same of the other bar.

The two Tables following contain these comparisons.

Comparisons of the two Yards of **O T** with Y_{55}.

Date. 1864.		Temperature.	O T		Y_{55}		Difference in Micrometer Divisions.		Equivalent.
			Yard [α·β].						
Sept.	19	60°·94	11·38 *h*	+26·42 *k*	20·17 *h*	+24·50 *k*	+8·79 *h*	−1·92 *k*	5·46
,,	,,	60·95	19·95 *h*	+17·07 *k*	19·57 *h*	+25·72 *k*	−0·38 *h*	+8·65 *k*	6·60
,,	,,	60·86	15·43 *h*	+22·98 *k*	19·45 *h*	+26·50 *k*	+4·02 *h*	+3·52 *k*	6·00
,,	20	60·20	26·80 *h*	+28·92 *k*	29·53 *h*	+34·35 *k*	+2·73 *h*	+5·43 *k*	6·50
,,	,,	60·20	28·00 *h*	+28·05 *k*	30·08 *h*	+34·03 *k*	+2·08 *h*	+5·98 *k*	6·43
,,	,,	60·13	27·45 *h*	+28·53 *k*	29·13 *h*	+33·67 *k*	+1·68 *h*	+5·14 *k*	5·44
,,	22	59·44	28·45 *h*	+33·22 *k*	27·75 *h*	+43·10 *k*	−0·70 *h*	+9·88 *k*	7·33
,,	23	59·50	31·08 *h*	+29·15 *k*	33·87 *h*	+34·97 *k*	+2·79 *h*	+5·82 *k*	6·86
,,	,,	59·51	28·03 *h*	+33·38 *k*	30·72 *h*	+38·42 *k*	+2·69 *h*	+5·04 *k*	6·16
,,	,,	59·53	19·80 *h*	+40·93 *k*	27·97 *h*	+40·60 *k*	+8·17 *h*	−0·33 *k*	6·23
			Yard [β·γ].						
,,	21	59·56	30·87 *h*	+33·45 *k*	32·33 *h*	+34·85 *k*	+1·46 *h*	+1·40 *k*	2·28
,,	,,	59·59	29·20 *h*	+35·28 *k*	33·20 *h*	+34·32 *k*	+4·00 *h*	−0·96 *k*	2·41
,,	,,	59·55	37·25 *h*	+27·97 *k*	37·23 *h*	+29·57 *k*	−0·02 *h*	+1·60 *k*	1·26
,,	22	59·34	33·98 *h*	+29·67 *k*	36·03 *h*	+33·87 *k*	+2·05 *h*	+4·20 *k*	4·98
,,	,,	59·42	29·28 *h*	+33·65 *k*	36·00 *h*	+34·15 *k*	+6·72 *h*	+0·50 *k*	5·74
,,	,,	59·48	32·20 *h*	+31·25 *k*	31·60 *h*	+36·65 *k*	−0·60 *h*	+5·40 *k*	3·83
,,	24	59·37	33·40 *h*	+32·55 *k*	32·10 *h*	+38·27 *k*	−1·30 *h*	+5·72 *k*	3·53
,,	,,	59·39	34·27 *h*	+31·72 *k*	33·03 *h*	+36·65 *k*	−1·24 *h*	+4·93 *k*	2·95
,,	,,	59·44	32·85 *h*	+32·88 *k*	35·08 *h*	+34·45 *k*	+2·23 *h*	+1·57 *k*	3·03
,,	,,	59·51	33·95 *h*	+30·63 *k*	33·75 *h*	+35·30 *k*	−0·20 *h*	+4·67 *k*	3·57

Temperatures corrected for errors of thermometers.

Comparisons of Space [γ·δ] on **O T** with Space [τ·*f*] on **O F**.

September 26th 1864.

Mean temperature 59°·80.

O T		O F		Difference in Micrometer Divisions.		[γ·δ]−[τ·*f*]
15·53 *h*	+24·40 *k*	22·00 *h*	+24·27 *k*	+6·47 *h*	−0·13 *k*	5·04
12·30 *h*	+28·17 *k*	12·13 *h*	+32·73 *k*	−0·17 *h*	+4·56 *k*	3·50
18·50 *h*	+21·80 *k*	20·33 *h*	+24·47 *k*	+1·83 *h*	+2·67 *k*	3·59
21·87 *h*	+19·23 *k*	21·83 *h*	+23·87 *k*	−0·04 *h*	+4·64 *k*	3·67
19·40 *h*	+21·77 *k*	24·03 *h*	+21·13 *k*	+4·63 *h*	−0·64 *k*	3·17
19·23 *h*	+21·33 *k*	19·80 *h*	+25·27 *k*	+0·57 *h*	+3·92 *k*	3·58
22·70 *h*	+18·10 *k*	19·67 *h*	+25·13 *k*	−3·03 *h*	+7·03 *k*	3·20
23·40 *h*	+17·97 *k*	24·03 *h*	+21·37 *k*	+0·63 *h*	+3·40 *k*	3·21
22·43 *h*	+18·47 *k*	24·13 *h*	+21·30 *k*	+1·70 *h*	+2·83 *k*	3·61
22·67 *h*	+17·80 *k*	23·07 *h*	+22·10 *k*	+0·40 *h*	+4·30 *k*	3·75

If we take the means of the second and last columns of the first Table we find that the yard $[\alpha\cdot\beta]$ was at the temperature of $60^\circ\cdot 13$, greater than the Standard Yard by $6\cdot 20$; and the yard $[\beta\cdot\gamma]$ at the temperature $59^\circ\cdot 46$, greater than the Standard Yard by $3\cdot 36$. Now the values we have obtained for these yards are, see page 102,

$$[\alpha\cdot\beta] = \mathbf{Y}_{55} + 6\cdot 18 - 0\cdot 442\,(t - 62)$$
$$[\beta\cdot\gamma] = \mathbf{Y}_{55} + 2\cdot 45 - 0\cdot 442\,(t - 62)$$

if here we make in the first equation $t = 60\cdot 13$ and in the second $t = 59\cdot 46$ we obtain

$$\text{at } 60^\circ\cdot 13\ [\alpha\cdot\beta] = \mathbf{Y}_{55} + 7\cdot 01$$
$$\text{at } 59^\circ\cdot 46\ [\beta\cdot\gamma] = \mathbf{Y}_{55} + 3\cdot 57$$

which are somewhat larger than the quantities given by the last series of comparisons, but not sufficiently different to indicate any real change of length.

We shall therefore combine these observations with the thirty comparisons of each yard given at pages 98, 99.

As before, let

$$[\alpha\cdot\beta] = \mathbf{Y}_{55} + x' + f'y$$
$$[\beta\cdot\gamma] = \mathbf{Y}_{55} + x_{,} + f_{,}y$$

Where f is the temperature of the bars $- 62^\circ$. Then we have for the determination of x' $x_{,}$ and y a series of eighty equations of the form

$$x' + f'y - a' = 0$$
$$\vdots$$
$$x_{,} + f_{,}y - a_{,} = 0$$
$$\vdots$$

which treated according to the method of least squares give

$$40\,x' + (f')\,y - (a') = 0 \qquad (1)$$
$$40\,x_{,} + (f_{,})\,y - (a_{,}) = 0$$
$$(f')\,x' + (f_{,})\,x_{,} + [(f'^2) + (f_{,}^2)]\,y - [(f'a') + (f_{,}a_{,})] = 0$$

The results of the multiplications here implied, are

$$(f') = -\ 288\cdot 19 \qquad (2)$$
$$(a') = 367\cdot 58$$
$$(a'f') = -\ 4220\cdot 5802$$
$$(f'^2) = 5676\cdot 1503$$
$$(f_{,}) = -\ 298\cdot 95$$
$$(a_{,}) = 227\cdot 98$$
$$(a_{,}f_{,}) = -\ 2906\cdot 3723$$
$$(f_{,}^2) = 4803\cdot 8443$$

And the final equations,

$$40\,x' - 288\cdot 19\,y - 367\cdot 58 = 0 \qquad (3)$$
$$40\,x_{,} - 298\cdot 95\,y - 227\cdot 98 = 0$$
$$-\ 288\cdot 19\,x' - 298\cdot 95\,x_{,} + 10479\cdot 99\,y + 7126\cdot 952 = 0$$

From which

$$x' = \quad 5 \cdot 949 \qquad (4)$$
$$x_, = \quad 2 \cdot 338$$
$$y = -0 \cdot 4498$$

If we write A B C in place of the absolute terms of the equations, we get

$$0 = x' + \cdot 0334139\,A + \cdot 0087280\,B + \cdot 0011678\,C \qquad (5)$$
$$0 = x_, + \cdot 0087280\,A + \cdot 0340539\,B + \cdot 0012114\,C$$
$$0 = y + \cdot 0011678\,A + \cdot 0012114\,B + \cdot 0001621\,C$$

Thus we have for the lengths of the two yards

$$[\alpha \cdot \beta] = \mathbf{Y}_{63} + 5 \cdot 95 - 0 \cdot 4498\,(t - 62) \qquad (6)$$
$$[\beta \cdot \gamma] = \mathbf{Y}_{55} + 2 \cdot 34 - 0 \cdot 4498\,(t - 62)$$

which are less than the values resulting from the first 30 comparisons of each yard by 0·23 and 0·11.

From the last column of the Table, page 140, it appears that at the temperature of 59°·80 $[\gamma \cdot \delta]$ is greater than the space $[\tau \cdot f]$ on **O F** by 3·63. Now the excess of the expansion of the former over the latter we have found, page 103, to be

$$\frac{474}{3600}\,y - \cdot 0026$$

where y is as found above $- \cdot 4498$; that is, $- \cdot 0618$ per 1° Fahrenheit. Consequently the correction to reduce observations at 59°·80 to 62°·00 is $- \cdot 0618 \times 2 \cdot 20 = -0 \cdot 13$. This is to be applied to the observed results, and if we correct (by $+ \cdot 03$), the observed differences taken at the temperature of 62°·53, recorded page 102, the general mean of the forty comparisons is 3·40, so that at 62°

$$[\gamma \cdot \delta] = [\tau \cdot f] + 3 \cdot 40 \qquad (7)$$

This differs but little from the result of the first 30 comparisons, viz., 3·37.

We shall now bring together the errors of the 40 comparisons of each space in the following Table :—

$[\alpha \cdot \beta]$		$[\beta \cdot \gamma]$		$[\gamma \cdot \delta]$	
−0·25	−1·68	−0·49	+0·43	+0·48	+0·04
+0·56	−0·46	−0·75	+0·17	+0·41	+0·72
−1·33	+1·75	+0·05	−0·01	−1·11	−0·57
−0·63	+0·80	+0·48	−0·72	+0·08	+0·00
−1·18	+0·82	+0·26	−0·61	+0·02	−0·61
+0·35	−0·04	+0·66	−0·15	−0·10	+0·04
+0·80	−0·05	−0·93	−0·31	−0·87	+0·13
−0·67	+0·54	−0·02	−0·18	+0·28	−0·99
+0·61	+0·23	−1·31	−0·47	+0·28	−0·03
−0·24	+0·28	−0·07	−1·04	−0·03	−0·27
−1·32	+0·97	+0·67	+1·16	+0·01	+1·51
−1·64	−0·18	+0·56	+1·01	+0·84	−0·03
+0·07	+0·46	+0·63	+2·18	−0·84	+0·06
−1·36	+0·26	+1·25	−1·44	−0·06	+0·14
+0·35	+0·33	−0·45	−2·24	−0·54	−0·36
−0·75	+1·35	+0·72	−0·36	+0·77	+0·05
+1·04	−0·23	+1·18	−0·01	+0·71	−0·33
+0·30	+0·21	−0·30	+0·56	−0·18	−0·32
−0·59	+0·91	−0·24	+0·46	+0·35	+0·08
−1·18	+0·83	−0·13	−0·11	+0·01	+0·22

The errors in the last series of comparisons of the yard $[\beta \cdot \gamma]$ are unusually large. Particular attention was drawn to the irregularity of the results while the observations were in progress, but no cause could be assigned, unless it might possibly be that the stone piers were of a slightly different temperature to that of the air of the room. It will be seen that the three first observations have positive errors while the next three have negative errors. In the first three the bar **O T** was next the piers, but in the next three $\mathbf{Y}_{55}$ was next the piers. The thermometer readings, however, do *not* indicate any change in the temperatures that would account for the apparent change of results. The cause of discrepancy, whatever it be, seems to vanish at the end of the series.

From equations (5) we find the reciprocals of the weights of the several determinations as follows:—

$$\begin{aligned} x' & \ldots\ldots 0 \cdot 03341 \\ x_{,} & \ldots\ldots 0 \cdot 03405 \\ x' + x_{,} & \ldots\ldots 0 \cdot 08492 \\ y & \ldots\ldots 0 \cdot 000162 \end{aligned} \qquad (8)$$

The sum of the squares of the residual errors of the 80 equations from which these quantities were obtained (being the errors in the first four columns of the last Table) is 54·65. Hence the probable error of a single comparison is

$$\pm \cdot 674 \sqrt{\frac{54 \cdot 65}{80 - 3}} = \pm 0 \cdot 568 \qquad (9)$$

The probable errors of the several determinations are therefore

$$\begin{aligned} x' & \ldots\ldots \pm 0 \cdot 568 \sqrt{\cdot 03341} = \pm 0 \cdot 104 \\ x_{,} & \ldots\ldots \pm 0 \cdot 568 \sqrt{\cdot 03405} = \pm 0 \cdot 105 \\ x' + x_{,} & \ldots\ldots \pm 0 \cdot 568 \sqrt{\cdot 08492} = \pm 0 \cdot 165 \\ y & \ldots\ldots \pm 0 \cdot 568 \sqrt{\cdot 000162} = \pm 0 \cdot 0072 \end{aligned} \qquad (10)$$

The sum of the squares of the 40 errors of the comparisons of $[\gamma \cdot \delta]$ is 10·51, consequently the probable error of a comparison is

$$\pm 0 \cdot 674 \sqrt{\frac{10 \cdot 51}{40 - 1}} = \pm 0 \cdot 350 \qquad (11)$$

and the probable error of the mean of the whole

$$\frac{\pm 0 \cdot 350}{\sqrt{40}} = \pm \cdot 055 \qquad (12)$$

We have therefore

$$[\gamma \cdot \delta] = [\tau \cdot f] + 3 \cdot 40 \pm 0 \cdot 055 \qquad (13)$$

Also, for $t = 62°$, see page 72.

$$[\tau \cdot f] = \frac{474}{3600} \mathbf{Y}_{55} - 3 \cdot 24 \pm 0 \cdot 127$$

therefore,

$$[\gamma \cdot \delta] = \frac{4 \cdot 74}{3600} \mathbf{Y}_{55} + 0 \cdot 16 \pm 0 \cdot 137 \qquad (14)$$

Adding to this the sum of equations (6), viz. :—

$$[\alpha \cdot \gamma] = 2\, \mathbf{Y}_{55} + 8\cdot 29 \pm 0\cdot 165 \qquad (15)$$

we get for the entire length of the bar—

$$[\alpha \cdot \delta] = \frac{7674}{3600}\, \mathbf{Y}_{55} + 8\cdot 45 \pm 0\cdot 214$$

so that the length of the Ordnance Toise in terms of $\mathbf{Y}_{55}$, both being at 62°, is

$$\mathbf{T}_{\circ} = (2\cdot 13167512 \pm \cdot 00000021)\, \mathbf{Y}_{55} \qquad (16)$$

2.

A series of measures of the space between the lines on the contact apparatus when the two parts are brought together was also made with the micrometers H and K. In each measure the focus was re-adjusted, and the contacts renewed. The results are given in the following Table, each number being the mean of three micrometer readings.

Micrometer H.			Micrometer K.		
Left Line.	Right Line.	Diff.	Left Line.	Right Line.	Diff.
1366·17	654·83	711·34	657·67	1365·53	707·86
1364·13	652·50	711·63	659·25	1366·57	707·30
1359·83	650·03	709·80	659·40	1368·00	708·60
1358·03	647·33	710·70	657·30	1366·13	708·83
1356·90	646·63	710·27	658·60	1366·50	707·90

The mean of the measures by H is 710·75, which in millionths of the yard = 564·97; and the mean of the measures by K is 708·10, which in millionths of a yard = 565·06. The mean of these, 565·04, may be taken, when compared with the value (17), page 133, otherwise obtained, as a satisfactory proof that the contact apparatus has not undergone any change. We shall not make any further use of this result, retaining the value given in the Section on the Prussian Toise.

3.

Rather unfortunately for the comparisons of the Toise No. 11 with the Ordnance Toise, the temperature commenced to fall just after September 29, falling 5° in the next week. Unfortunately, inasmuch as we have to make use of the relative expansion of the two bars to reduce our results to 62°. If τ_{10} τ_{11} be the expansions per 1° Fahrenheit of the Toises Nos. 10 and 11, τ_0 that of the Ordnance Toise

$$(\tau_{11} - \tau_0) = (\tau_{10} - \tau_0) - \tau_{10} + \tau_{11}$$

Now from page 49 of the "*Compte rendu des opérations . . . . . pour étalonner les règles qui ont été employées à la mesure des bases géodésiques Belges*," it appears that the expansions for 1° Centigrade of the Toises Nos. 10 and 11 are $0^{t}\cdot 009128$ and $0^{t}\cdot 009156$ respectively; that is for 1° Fahrenheit, the expansions expressed in millionths of a yard are

No. 10 . . . . . 12·511
No. 11 . . . . . 12·549

Let the probable errors of these quantities (not known) be ϵ_{10} ϵ_{11}. By page 130 we see that

$$\tau_{10} - \tau_0 = 0\cdot 268 \pm 0\cdot 008$$

Then if τ_{10} and τ_{11} had been independently determined

$$\tau_{11} - \tau_0 = 0\cdot 306 \pm \sqrt{\epsilon_{10}^2 + \epsilon_{11}^2 + (\cdot 008)^2}$$

Twenty-five comparisons were obtained on September 30 and October 1, 3, 4, 5, 6, 7, 8, at which time the temperature had fallen to 53°·9. As it was very desirable from the shortness of the time in which the comparisons were to be made, to have all the circumstances varied as much as possible, the bars were frequently dismounted in the evenings and the adjustments renewed previous to the next morning's observations.

In order to ensure the non-existence of a constant error in a series of observations of this kind, the following points require to be attended to:—

1. The axes of rotation of the microscopes to be vertical.
2. The outer foci of the microscopes to be in a horizontal line.
3. Ordnance Toise to rest symmetrically on its rollers.
4. The supporting rollers of the Belgian Toise to be in a straight line.
5. The toise to lie symmetrically on them.
6. The steel needles of the contact apparatus to be horizontal.
7. Point of contact to be on the horizontal diameter of the circular terminal disk of the toise.
8. Point of contact to be on the vertical diameter of the same.
9. Each piece of the contact apparatus to be level transversely.
10. Care to be taken that no partical of dust intervenes at the contacts.
11. The bars to be adjusted to distinct focus.
12. The bars to take alternately the inner and outer positions.

If any of these adjustments should remain unaltered during the series, there might be room to fear a possible constant error; and on the other hand the more frequently they are renewed the more confidence may be placed in the results. If it were practicable it would be well that they were all renewed previous to each comparison, but the time that would be required for this purpose renders it impracticable; besides which the mere presence of the observer for the length of time requisite for correcting many adjustments produces a considerable disturbance in the length of the bars, so that they require to be left alone for many hours to settle to their natural lengths.

The manner in which the adjustments were actually renewed, will be seen from the notes following the Table subjoined, which contains the results of the twenty-five comparisons.

No.	Date.	Temp.	Ordnance Toise.	Belgian Toise.	Difference in Micrometer Divisions.	$T_{11} - T_0 + \sigma$
	1864:	°				
1	Sept. 30	58·94	1130·54 *h* + 1129·00 *k*	874·82 *h* + 871·23 *k*	255·72 *h* + 257·77 *k*	408·97
2	" "	58·92	1128·27 *h* + 1131·50 *k*	874·07 *h* + 872·70 *k*	254·20 *h* + 258·80 *k*	408·59
3	Oct. 1	58·41	1132·75 *h* + 1129·95 *k*	875·43 *h* + 873·84 *k*	257·32 *h* + 256·11 *k*	408·92
4	" "	58·43	1130·77 *h* + 1132·57 *k*	873·55 *h* + 876·02 *k*	257·22 *h* + 256·55 *k*	409·19
5	" "	58·37	1130·12 *h* + 1133·58 *k*	875·08 *h* + 875·55 *k*	255·04 *h* + 258·03 *k*	408·64
6	" 3	57·04	1134·80 *h* + 1135·30 *k*	878·03 *h* + 879·22 *k*	256·77 *h* + 256·08 *k*	408·46
7	" "	57·01	1135·97 *h* + 1134·55 *k*	878·80 *h* + 877·63 *k*	257·17 *h* + 256·92 *k*	409·45
8	" "	57·01	1137·18 *h* + 1135·02 *k*	877·63 *h* + 879·77 *k*	259·55 *h* + 255·25 *k*	410·01
9	" "	57·02	1138·07 *h* + 1133·10 *k*	880·27 *h* + 877·28 *k*	257·80 *h* + 255·82 *k*	409·07
10	" 4	56·26	1138·82 *h* + 1137·88 *k*	882·63 *h* + 883·57 *k*	256·19 *h* + 254·31 *k*	406·58
11	" "	56·27	1137·92 *h* + 1138·47 *k*	883·70 *h* + 882·93 *k*	254·22 *h* + 255·54 *k*	406·00
12	" "	56·20	1136·38 *h* + 1139·92 *k*	886·18 *h* + 879·87 *k*	250·20 *h* + 260·05 *k*	406·40
13	" "	56·15	1136·40 *h* + 1139·25 *k*	882·88 *h* + 882·72 *k*	253·52 *h* + 256·53 *k*	406·23
14	" 5	55·16	1144·63 *h* + 1139·42 *k*	887·55 *h* + 886·15 *k*	257·08 *h* + 253·27 *k*	406·46
15	" "	55·26	1141·05 *h* + 1142·02 *k*	885·75 *h* + 884·93 *k*	255·30 *h* + 257·09 *k*	408·10
16	" "	55·11	1142·77 *h* + 1142·33 *k*	886·12 *h* + 887·57 *k*	256·65 *h* + 254·76 *k*	407·31
17	" "	55·10	1141·98 *h* + 1143·28 *k*	888·12 *h* + 885·30 *k*	253·86 *h* + 257·98 *k*	407·66
18	" 6	54·39	1143·38 *h* + 1142·47 *k*	886·20 *h* + 887·32 *k*	257·18 *h* + 254·65 *k*	407·64
19	" "	54·36	1145·03 *h* + 1141·25 *k*	887·45 *h* + 886·05 *k*	257·58 *h* + 255·20 *k*	408·40
20	" "	54·28	1142·05 *h* + 1145·33 *k*	886·77 *h* + 888·45 *k*	255·28 *h* + 256·88 *k*	407·91
21	" 7	53·89	1147·20 *h* + 1142·57 *k*	888·17 *h* + 891·08 *k*	259·03 *h* + 251·49 *k*	406·59
22	" "	53·94	1148·35 *h* + 1141·47 *k*	888·90 *h* + 890·00 *k*	259·45 *h* + 251·47 *k*	406·91
23	" "	54·09	1144·12 *h* + 1144·30 *k*	889·27 *h* + 887·20 *k*	254·85 *h* + 257·10 *k*	407·75
24	" 8	53·88	1127·37 *h* + 1129·28 *k*	872·02 *h* + 872·85 *k*	255·35 *h* + 256·43 *k*	407·61
25	" "	53·92	1128·30 *h* + 1128·33 *k*	871·43 *h* + 873·08 *k*	256·87 *h* + 255·25 *k*	407·88

Temperatures corrected for errors of thermometers.

(1.) Bars re-adjusted to focus. Contacts renewed.
(3.) Bars re-adjusted to focus. Contacts renewed.
(5.) Microscopes re-levelled and re-adjusted. Contact pieces re-levelled, re-adjusted.
(6.) Bars re-adjusted to focus. Contacts renewed.
(7.) Bars re-adjusted to focus.
(8.) Bars re-adjusted to focus. Contacts renewed.
(9.) Bars dismounted and interchanged. Rollers of Belgian Toise re-adjusted into line. Bars re-adjusted to focus. Contacts renewed.
(10.) Contact pieces re-levelled. Focus of bars re-adjusted.
(11.) Bars re-adjusted to focus.
(12.) Microscopes re-levelled.
(13.) Bars dismounted and interchanged. Rollers of Belgian Toise re-adjusted into line. Contact pieces re-levelled and adjusted transversely. Bars re-adjusted to focus.
(14.) Bars re-adjusted to focus. Contact pieces re-levelled and adjusted in vertical height.
(15.) Bars re-adjusted to focus.
(16.) Bars re-adjusted to focus.
(17.) Bars dismounted and interchanged. Contact pieces levelled and adjusted transversely. Bars adjusted to focus.
(18.) Bars re-adjusted to focus.
(19.) Contacts renewed.
(20.) Bars dismounted and interchanged. Rollers of Belgian Toise re-adjusted into line. Contact pieces re-adjusted and levelled.
(21.) Bars re-adjusted to focus.
(22.) Bars re-adjusted to focus.
(23.) Microscopes re-levelled. Focus re-adjusted and contacts renewed.

The several adjustments here enumerated were made at the close of the comparisons against which they are written. In renewing the contacts, the steel needle of each piece was drawn back (against its spring) from the bar, and both the contact piece and the end of the Toise very gently rubbed with a piece of a fine kid glove.

The following Table shows the corrections for temperature, taking 0·306 as the excess of the expansion of the Belgian Toise above the Ordnance Toise for 1° Fahrenheit.

Observed differences of Length.	Correction for Temperature.	Difference of Length at 62°.	Errors.
408·97	0·94	409·91	+0·20
408·59	0·94	409·53	−0·18
408·92	1·10	410·02	+0·31
409·19	1·09	410·28	+0·57
408·64	1·11	409·75	+0·04
408·46	1·52	409·98	+0·27
409·45	1·53	410·98	+1·27
410·01	1·53	411·54	+1·83
409·07	1·52	410·59	+0·88
406·58	1·76	408·34	−1·37
406·00	1·75	407·75	−1·96
406·40	1·77	408·17	−1·54
406·23	1·79	408·02	−1·69
406·46	2·09	408·55	−1·16
408·10	2·06	410·16	+0·45
407·31	2·11	409·42	−0·29
407·66	2·11	409·77	+0·06
407·64	2·33	409·97	+0·26
408·40	2·34	410·74	+1·03
407·91	2·36	410·27	+0·56
406·59	2·48	409·07	−0·64
406·91	2·47	409·38	−0·33
407·75	2·42	410·17	+0·46
407·61	2·48	410·09	+0·38
407·88	2·47	410·35	+0·64

4.

The mean of the corrected differences of length is 409·71 and the sum of the squares of the differences of the individuals with the mean result is 21·612 which would give the probable error of a single observation or comparison as ±0·640. But this is greater than the actual probable error of a comparison, as it is affected by the probable error of the assumed rate of relative expansion.

But in consequence of some of these comparisons being further from the normal temperature than others, it is not correct to take the simple mean of the third column as the final result. If we denote by a the observed differences of length, the differences of the corresponding temperatures from 62° by α, and the difference of expansions of the Belgian Toise and Ordnance Toise by x; then the results for the difference of length y at 62° are

$$\begin{aligned} y &= a_1 + \alpha_1 x \qquad (17) \\ y &= a_2 + \alpha_2 x \\ y &= a_3 + \alpha_3 x \\ &\vdots \\ y &= a_n + \alpha_n x. \end{aligned}$$

In order to obtain the most probable value of y we shall multiply these values by a series of multipliers $\lambda_1\, \lambda_2\, \ldots\ldots\, \lambda_n$ such as will give a resulting value of y with the minimum probable error: the sum of such multipliers to be unity. If E_i be the actual error of the measured quantity a_i and E_0 the actual error of the assumed relative expansion x, then the error of the adopted result for y is

$$\lambda_1\, E_1 + \lambda_2\, E_2 + \ldots\ldots \lambda_n\, E_n + (\alpha_1\, \lambda_1 + a_2\, \lambda_2 + \ldots\, \alpha_n\, \lambda_n)\, E_0 \qquad (18)$$

Write V for the coefficient of E_0 so that

$$V = \alpha_1\, \lambda_1 + \alpha_2\, \lambda_2 + \ldots\ldots \alpha_n\, \lambda_n \qquad (19)$$

then the square of probable error of the adopted result for y will be

$$e^2\, (\lambda_1^2 + \lambda_2^2 + \ldots\, \lambda_n^2) + V^2\, \epsilon^2 \qquad (20)$$

where e is the probable error of a single comparison, ϵ the probable error of our assumed rate of relative expansion. Now this quantity is to be a minimum, subject to the condition

$$\lambda_1 + \lambda_2 + \lambda_3 + \ldots\, \lambda_n = 1 \qquad (21)$$

Differentiating these two equations, we have

$$0 = (e^2\, \lambda_1 + \epsilon^2\, V\, \alpha_1)\, d\,\lambda_1 + (e^2\, \lambda_2 + \epsilon^2\, V\, \alpha_2)\, d\,\lambda_2 + \ldots\ldots (e^2\, \lambda_n + \epsilon^2\, V\, \alpha_n)\, d\,\lambda_n$$

$$0 = d\,\lambda_1 + d\,\lambda_2 \ldots\ldots\ldots\ldots + d\,\lambda_n$$

from which

$$\begin{aligned} e^2\, \lambda_1 + \epsilon^2\, V\, \alpha_1 &= C \\ e^2\, \lambda_2 + \epsilon^2\, V\, \alpha_2 &= C \\ e^2\, \lambda_3 + \epsilon^2\, V\, \alpha_3 &= C \\ &\vdots \\ e^2\, \lambda_n + \epsilon^2\, V\, \alpha_n &= C \end{aligned} \qquad (22)$$

The sum of these equations gives

$$e^2 + \epsilon^2\, V\, (\alpha) = n\, C \qquad (23)$$

and if we multiply them by $\alpha_1\, \alpha_2 \ldots\ldots \alpha_n$ respectively, and add, we get

$$e^2\, V + \epsilon^2\, V\, (\alpha^2) = C\, (\alpha) \qquad (24)$$

eliminating C, we get

$$V = \frac{\dfrac{(\alpha)}{n}}{1 + \dfrac{\epsilon^2}{e^2}\left[(\alpha^2) - \dfrac{(\alpha)^2}{n}\right]} \qquad (25)$$

and further

$$\begin{aligned} \lambda_1 &= \frac{1}{n} + \frac{\epsilon^2}{e^2}\left[\frac{(\alpha)}{n} - \alpha_1\right] V \\ \lambda_2 &= \frac{1}{n} + \frac{\epsilon^2}{e^2}\left[\frac{(\alpha)}{n} - \alpha_2\right] V \\ \lambda_3 &= \frac{1}{n} + \frac{\epsilon^2}{e^2}\left[\frac{(\alpha)}{n} - \alpha_3\right] V \\ &\vdots \\ \lambda_n &= \frac{1}{n} + \frac{\epsilon^2}{e^2}\left[\frac{(\alpha)}{n} - \alpha_n\right] V \end{aligned} \qquad (26)$$

These are the multipliers which render the probable error a minimum. The sum of their squares is

$$\lambda_1^2 + \lambda_2^2 + \cdots \lambda_n^2 = \frac{1}{n} + \left[(\alpha^2) - \frac{(\alpha)^2}{n}\right]\frac{\epsilon^4}{e^4}V^2$$

Hence for the square of the probable error we have

$$\frac{e^2}{n} + \epsilon^2\left[(\alpha^2) - \frac{(\alpha)^2}{n}\right]\frac{\epsilon^2}{e^2}V^2 + \epsilon^2 V^2$$

$$= \frac{e^2}{n} + \epsilon^2 V^2 \left\{1 + \frac{\epsilon^2}{e^2}\left[(\alpha^2) - \frac{(\alpha)^2}{n}\right]\right\}$$

$$= \frac{e^2}{n} + \frac{\epsilon^2 \dfrac{(\alpha)^2}{n^2}}{1 + \dfrac{\epsilon^2}{e^2}\left[(\alpha^2) - \dfrac{(\alpha)^2}{n}\right]} \qquad (27)$$

And this is a smaller quantity than would result from any other system of multipliers.

Now the quantity ϵ is not given. We may, however, get an estimate of its amount probably not very far from the truth, by an examination of the probable errors of the determinations of expansions of other bars. Referring to a previous section, we have obtained for the expansion of the bar **IO_1** the value $21 \cdot 055 \pm \cdot 089$, and for **$IO_2$** the value $21 \cdot 400 \pm 050$ (see page 79). Now in this last case, the probable error of the determination is about $\frac{1}{400}$th part of the amount of the expansion. Again, at page 49 of the Account of the Russian Meridian Arc, Vol. I., we have the probable errors of the determinations of the expansions of the four Standard Bars **N**, **P**, **F**, **H**, and taking the *mean* we get this, that the probable error of the determined amount of expansion is $\frac{1}{404}$th of that amount. Taking this as a guide, as the expansion of either toise is about $12 \cdot 5$, the probable error of the determination would be $\pm \cdot 03$, and if the determinations of the expansions of the two toises were independently made the probable error of the relative expansion would be $\sqrt{(\cdot 03)^2 + (\cdot 03)^2} = \pm \cdot 04$. In adopting this value we shall not err very materially from the truth; that is, not so much as to be misguided.

If further we take for the value of $e \pm 0 \cdot 60$, then

$$\frac{\epsilon^2}{e^2} = \frac{1}{225} \qquad (28)$$

This gives for the value of V

$$V = 4 \cdot 577$$

Forming now the values of λ and multiplying each result by the corresponding multiplier, we have finally

$$y = 409 \cdot 75 \qquad (29)$$

differing but very slightly from the mean, which was found to be $409 \cdot 71$.

By equation (27) the probable error of our result is

$$\pm \frac{e}{\sqrt{n}}\left\{1 + \frac{\dfrac{1}{225}\,\dfrac{(\alpha)^2}{25}}{1 + \dfrac{1}{225}\left[(\alpha^2) - \dfrac{(\alpha)^2}{n}\right]}\right\}^{\frac{1}{2}} = \pm \cdot 40\, e$$

Now we have seen that e is something less than $\pm \cdot 64$, so that our probable result may be set down as $\pm \cdot 25$.

Hence

$$\mathbf{T}_{11} = \mathbf{T}_0 - \sigma + 409 \cdot 75 \pm 0 \cdot 25 \tag{30}$$

Where $\mathbf{T}_{11}$ is the length of the Belgian Toise at 62° Fahr., $\mathbf{T}_o$ that of the Ordnance Toise, and σ the value of the space on the contact apparatus.

5.

The value of σ is given at page 134 as

$$\sigma = 565 \cdot 85 \pm \cdot 108$$

and for $\mathbf{T}_0$ we have found

$$\mathbf{T}_0 = (2 \cdot 13167512 \pm 000000214)\ \mathbf{Y}_{65}$$

consequently

$$\mathbf{T}_{11} = 2 \cdot 13151902\ \mathbf{Y}_3$$

with a probable error, to be regarded only as *approximate*, of

$$\sqrt{(\cdot 214)^2 + (\cdot 108)^2 + (\cdot 25)^2} = \pm 0 \cdot 35$$

Therefore, finally, at the temperature of 62° we have the length of the Belgian Toise

$$\mathbf{T}_{11} = (2 \cdot 13151902 \pm \cdot 00000035)\ \mathbf{Y}_{65} \tag{31}$$

XIII.

COMPARISONS OF STANDARD YARDS.

1.

The copies of the National Standard Yard assigned to the Ordnance Survey are—

No. 29 Bronze.

No. 55 Swedish Iron **B**.

It is stated in the official report that the temperatures at which they are exactly one yard, are as follows:—

No. 29 at $61^\circ \cdot 51$ = National Standard Yard.

No. 55 at $61^\circ \cdot 93$ = National Standard Yard.

these values being the results of the comparisons made by the late Mr. Sheepshanks.

Of these two bars, the iron bar No. 55 appears to be best determined. According to information kindly supplied by the Astronomer Royal, No. 55 was compared directly with Bronze No. 28 (*the* Standard) on March 21, 22, August 10, 11, 12, 13, 18, 1853; Bronze 29 was compared on five days; some of the comparisons were with Bronze 28 direct, and some indirectly through another bar, Bronze No. 12.

That a standard bar does retain an invariable length at a stated temperature is a proposition not altogether established, though it may perhaps be said that in very few cases have actual suspected alterations or fluctuations of length been sufficiently proved. In the very bar chosen by Mr. Sheepshanks as the representative of the national unit, we have, for instance, strong evidence of a temporary alteration of length. In the ten feet standard of the Ordnance Survey, there is much reason to doubt the constancy of the coefficient of expansion, and if this be uncertain, it cannot be said that the length of the bar at 62° is invariable. In the Report* of the Comparisons of Standard Bars in Brussels, 1854, we find similar instances of apparent changes both of length and of rates of expansion, pp. 105, 106, 126.

How far these discrepancies may be due to errors of observation and faulty methods of comparison, it is not easy to say; but the fact of their having been observed at all renders it very desirable to re-compare some of the Copies of the Standard Yard, with a view of ascertaining whether the differences obtained at present agree with those assigned by Mr. Sheepshanks ten years ago.

* "Compte rendu des Opérations de la Commission instituée par M. le Ministre de la Guerre, pour étalonner les règles, &c." Bruxelles, 1855.

Three copies were accordingly borrowed from the Astronomer Royal. They are designated—

No. 65. Cast Steel **A**
No. 66. Cast Steel **B**
No. 67. Cast Steel **C**

and their lengths as assigned by Mr. Sheepshanks—

No. 65 at $62^\circ \cdot 22$ = National Standard Yard.
No. 66 at $62^\circ \cdot 11$ = National Standard Yard.
No. 67 at $62^\circ \cdot 47$ = National Standard Yard.

Of these copies, No. 66 appears to be the best determined; it was compared (according to information supplied by Mr. Airy) on 13 days with Bronze 28 direct. No. 65 was compared on one day only, and 67 on two days; in both these cases the comparisons were not directly with Bronze 28 but indirectly through an iron bar (Low Moor **B**) whose length was well determined.

Referring to the "Account of the Construction of the new National Standard of Length and its Principal Copies," we find that the excess of the lengths of Cast Steel **B** and Swedish **B** above Bronze 28 are (p. 57)

$$\text{No. 66. Cast Steel B} = \text{Bronze } 28 - 0^r \cdot 006565 - 0^r \cdot 03730370\,(t - 62)$$
$$\text{No. 55. Swedish Iron B} = \text{Bronze } 28 + 0 \cdot 004267 - 0 \cdot 03350280\,(t - 62)$$
$$\text{where } 1^r = 0^{\text{in.}} \cdot 003587$$

In the same place we find

$$\text{Brass } 2 = \text{Bronze } 28 + 0^r \cdot 082740 + 0^r \cdot 00072975\,(t - 62)$$

It does not appear from the "Account" that the *absolute* expansion of Bronze 28 was ever determined. It appears to have been assumed to be the same as that of another Bronze Bar No. 12, of which the absolute expansion was determined by direct experiment. But we have the absolute expansion of the bar designated Brass 2 by direct experiment (pages 48, 49), it was found to be

$$0^r \cdot 09601 \ : \text{ for } 1^\circ \text{ Fahrenheit.}$$

From this and from the relative expansions shown above with reference to Bronze 28, we get for the expansion of Cast Steel **B**, 0·057977, and for Swedish Iron **B**, 0·061777, so that the absolute lengths of these bars at the temperature t are, if we put $\mathfrak{Y}$ to represent a TRUE YARD,

$$\text{No. 66. . . Cast Steel } \mathbf{B} = \mathfrak{Y} - 0^r \cdot 006565 + 0^r \cdot 057977\,(t - 62)$$
$$\text{No. 55. Swedish Iron } \mathbf{B} = \mathfrak{Y} + 0 \cdot 004267 + 0 \cdot 061777\,(t - 62)$$

or, if we express the small quantities in millionths of the yard—

$$\text{No. 66. . . Cast Steel } \mathbf{B} = \mathfrak{Y} - 0 \cdot 65 + 5 \cdot 7768\,(t - 62) = \mathbf{Y}_{60}$$
$$\text{No. 55. Swedish Iron } \mathbf{B} = \mathfrak{Y} + 0 \cdot 43 + 6 \cdot 1555\,(t - 62) = \mathbf{Y}_{55}$$

We find at page 59, that at 62° the length of Bronze 29 is in excess 4·60 divisions; at page 60, that Cast Steel **A** at 62° is in excess − 1·28 divisions, and Cast Steel **C** in excess by − 2·70 divisions. Expressing these differences in millionths of a yard, we have for the lengths at 62°

$$\text{No. 65. Cast Steel } \mathbf{A} = \mathfrak{Y} - 1 \cdot 28 = \mathbf{Y}_{65}$$
$$\text{No. 67. Cast Steel } \mathbf{C} = \mathfrak{Y} - 2 \cdot 69 = \mathbf{Y}_{67}$$
$$\text{No. 29. Bronze } = \mathfrak{Y} + 4 \cdot 58 = \mathbf{Y}_{29}$$

2.

In order to determine not only the present differences of length of these five bars but their differences of expansion, observations were made upon them, both at low temperatures, 38° to 40°, and in the vicinity, as near as practicable, of the normal temperature of 62°.

The three bars, 66, 55, 67, were compared on two days in February, 26th, 27th, being visited three times each day. The three bars, 65, 55, 29, were compared on February 29th and March 1, three times on the former and twice on the latter day. The order of observation and number of micrometer readings were as indicated in the following Table:—

Bar.	Number of readings of	
	H	K
No. 66	3	3
" 55	3	3
" 67	3	3
" 67	3	3
" 55	3	3
" 66	3	3

besides the reading of four thermometers in the bars at the commencement and close of the visit.

In each case the three bars were placed (in line) in the box of the ten feet bars. Each bar rested on two freely-moving rollers, which were capable of being raised or lowered by small quantities for accurate focal adjustment. These supporting rollers were in each case nine inches right and left of the centre of the bar. As indicated in the above Table, the three bars were brought under the microscopes successively, and observed in the direct order, α, β, γ, and then a second time in the reverse order, γ, β, α.

The long box not being suited to these comparisons, a box of mahogany was specially made, before the comparisons at the normal temperature, for the reception of three standard yards lying parallel to one another, and with their centres opposite. The distance apart of the outer yards was 2·80 inches, which left sufficient room for a third yard to lie between these. Each yard was supported on a pair of rollers 18 inches apart. In order to secure for the bars perfect freedom from any constraint, the rollers were two inches in diameter, and revolved very freely on their axles, which were of steel working in steel **Y**'s. By a special construction each of these rollers admitted of being moved in a strictly vertical plane by means of a slow motion screw working from below the box. Thus the observer was enabled to make his focal adjustments with the greatest nicety and the greatest ease. The cover of the box was provided with apertures for the proper reading of the bars and thermometers. The bars were laid on a slope towards the light, that is, their upper surfaces were inclined at an angle $\tan^{-1}\frac{1}{3}$ away from the observer; the rollers being in the form of frustra of cones, and having flanges to prevent the bars slipping off.

The inner and the outer bar had each two thermometers, which were read as usual at the commencement and close of every visit. The inner and outer bars were systematically interchanged, so that each was as often in the one position as in the other: thus, any difference of temperature owing to position is eliminated in the mean. Further, every comparison is affected with a different error, if it be sensible, of focal adjustment. The comparisons at 62° were conducted in exactly the same manner as described for those at the low temperature, with this exception, that two readings only of each microscope were taken in place of three, so that the small Table on page 153 will apply to these comparisons if we alter the numbers 3 into 2.

The bars 66, 55, 67, were compared on the following days, July 8, 9, 11, 12, 13, 14, 15, 22, 23: twenty comparisons or visits. The bars 65, 55, 29, were compared on July 15, 16, 18, 19, 20, 21, 22: twenty comparisons.

In the following Tables the results are given; each line being the result of one comparison. The first column contains the date; the second contains the mean of eight thermometer readings corrected for the errors of the thermometers; the third, fourth, and fifth contain the micrometer measures, expressed in micrometer divisions, h and k being the values of one division in the micrometer microscopes H and K respectively.

COMPARISONS OF STANDARD YARDS.

Nos. 66, 55, 67.

Date.		Mean Temperature.	Y_{66}	Y_{55}	Y_{67}
1864.		°			
February	26	38·26	$9·65\,h + 15·76\,k$	$13·46\,h + 22·88\,k$	$6·72\,h + 19·10\,k$
,,	,,	38·34	$19·56\,h + 4·79\,k$	$21·50\,h + 12·45\,k$	$15·80\,h + 7·58\,k$
,,	,,	38·43	$12·45\,h + 10·95\,k$	$17·82\,h + 15·18\,k$	$17·90\,h + 5·28\,k$
,,	27	38·38	$12·88\,h + 11·00\,k$	$14·80\,h + 18·66\,k$	$15·15\,h + 8·81\,k$
,,	,,	38·54	$11·15\,h + 13·92\,k$	$18·00\,h + 15·52\,k$	$8·35\,h + 16·70\,k$
,,	,,	38·63	$10·20\,h + 13·68\,k$	$9·30\,h + 24·03\,k$	$14·43\,h + 9·95\,k$
July	8	62·59	$21·08\,h + 20·80\,k$	$21·65\,h + 21·98\,k$	$22·85\,h + 20·30\,k$
,,	,,	62·71	$17·68\,h + 22·10\,k$	$20·63\,h + 18·60\,k$	$20·03\,h + 20·88\,k$
,,	,,	62·89	$19·40\,h + 19·43\,k$	$21·50\,h + 17·40\,k$	$19·50\,h + 20·75\,k$
,,	,,	63·04	$20·60\,h + 17·43\,k$	$18·08\,h + 20·40\,k$	$18·65\,h + 21·50\,k$
,,	9	62·90	$20·08\,h + 18·20\,k$	$21·83\,h + 18·03\,k$	$21·15\,h + 20·28\,k$
,,	,,	62·99	$18·43\,h + 20·63\,k$	$18·15\,h + 20·93\,k$	$18·03\,h + 21·93\,k$
,,	11	62·77	$17·88\,h + 23·78\,k$	$16·98\,h + 24·18\,k$	$23·08\,h + 18·78\,k$
,,	,,	63·17	$19·95\,h + 19·10\,k$	$19·90\,h + 19·50\,k$	$19·80\,h + 19·55\,k$
,,	,,	63·39	$17·85\,h + 20·75\,k$	$12·08\,h + 25·15\,k$	$14·48\,h + 25·13\,k$
,,	12	62·41	$19·80\,h + 22·23\,k$	$18·90\,h + 22·53\,k$	$20·90\,h + 22·83\,k$
,,	,,	62·49	$27·75\,h + 29·48\,k$	$26·93\,h + 32·10\,k$	$26·95\,h + 30·10\,k$
,,	,,	62·82	$24·65\,h + 31·60\,k$	$27·60\,h + 28·80\,k$	$29·22\,h + 26·98\,k$
,,	13	62·13	$31·58\,h + 27·45\,k$	$28·15\,h + 30·95\,k$	$29·18\,h + 29·95\,k$
,,	,,	62·29	$30·70\,h + 27·15\,k$	$27·55\,h + 31·23\,k$	$28·43\,h + 30·83\,k$
,,	14	61·82	$35·28\,h + 26·15\,k$	$35·03\,h + 26·35\,k$	$35·73\,h + 25·80\,k$
,,	,,	62·07	$30·48\,h + 28·25\,k$	$27·70\,h + 31·43\,k$	$27·95\,h + 30·95\,k$
,,	,,	62·39	$24·38\,h + 31·68\,k$	$31·33\,h + 25·85\,k$	$26·85\,h + 31·10\,k$
,,	15	61·42	$30·43\,h + 31·88\,k$	$32·30\,h + 30·20\,k$	$32·13\,h + 30·78\,k$
,,	22	66·02	$35·38\,h + 33·93\,k$	$34·53\,h + 33·32\,k$	$36·33\,h + 32·82\,k$
,,	23	65·63	$36·93\,h + 32·92\,k$	$34·83\,h + 34·65\,k$	$36·88\,h + 32·40\,k$

Temperatures corrected for errors of thermometers.

Comparisons of Standard Yards.

Nos. 65, 55, 29.

Date.		Mean Temperature.	Y_{65}	Y_{55}	Y_{29}
1864.		°			
February	29	39·21	9·00 h + 9·40 k	9·52 h + 19·53 k	50·96 h + 74·88 k
,,	,,	39·52	9·73 h + 8·45 k	13·98 h + 13·40 k	57·53 h + 65·85 k
,,	,,	39·73	8·26 h + 7·62 k	11·90 h + 13·88 k	55·63 h + 65·28 k
March	1	40·02	7·46 h + 8·05 k	8·20 h + 16·65 k	61·90 h + 55·20 k
,,	,,	40·23	1·63 h + 12·17 k	14·18 h + 8·87 k	69·00 h + 47·87 k
July	15	62·50	29·03 h + 28·08 k	27·48 h + 28·05 k	22·73 h + 28·73 k
,,	16	61·97	28·98 h + 32·03 k	27·20 h + 32·03 k	27·33 h + 29·75 k
,,	,,	62·71	29·83 h + 28·05 k	29·55 h + 26·73 k	22·33 h + 28·33 k
,,	,,	63·10	27·00 h + 28·75 k	25·70 h + 28·25 k	20·15 h + 26·33 k
,,	18	63·78	23·38 h + 29·78 k	19·80 h + 31·00 k	18·50 h + 22·88 k
,,	,,	64·20	25·10 h + 26·33 k	20·90 h + 28·28 k	19·55 h + 18·45 k
,,	,,	64·54	20·73 h + 27·50 k	22·90 h + 25·15 k	15·10 h + 19·33 k
,,	,,	64·82	27·10 h + 21·70 k	26·20 h + 19·55 k	13·63 h + 16·85 k
,,	19	64·91	22·13 h + 24·75 k	20·03 h + 25·08 k	14·45 h + 16·25 k
,,	,,	65·19	22·65 h + 24·18 k	22·38 h + 23·27 k	11·05 h + 17·48 k
,,	20	65·77	30·18 h + 40·17 k	29·45 h + 40·43 k	24·95 h + 24·90 k
,,	,,	66·05	36·73 h + 31·23 k	35·98 h + 32·48 k	25·90 h + 21·68 k
,,	,,	66·26	34·88 h + 32·27 k	36·85 h + 29·58 k	21·08 h + 25·52 k
,,	,,	66·53	38·18 h + 27·70 k	37·83 h + 26·62 k	20·65 h + 20·73 k
,,	21	65·33	32·28 h + 39·70 k	34·95 h + 38·15 k	25·60 h + 28·63 k
,,	,,	65·74	31·00 h + 39·23 k	31·13 h + 37·38 k	22·68 h + 25·10 k
,,	,,	65·97	31·10 h + 38·25 k	29·98 h + 38·60 k	22·45 h + 25·58 k
,,	,,	66·21	37·80 h + 29·43 k	37·98 h + 28·57 k	23·00 h + 21·50 k
,,	22	65·78	34·03 h + 35·12 k	32·70 h + 36·70 k	26·05 h + 23·68 k
,,	,,	65·97	27·50 h + 40·58 k	27·20 h + 39·80 k	14·00 h + 33·58 k

3.

To reduce these observations by the method of least squares we proceed as follows: Let the excesses of the lengths of the two bars being compared with Y_{55} be, at the temperature $62° + f$, with reference to that bar

$$x + fy$$
$$x' + fy'$$

or, if, Y_{55} be the length of No. 55 at the temperature $62° + f$ let the lengths of the other two be

$$Y_{55} + x + fy \qquad (1)$$
$$Y_{55} + x' + fy'$$

and let the distance of the zeros of the microscopes at the time of this comparison be

$$Y_{55} + z \qquad (2)$$

Let the quantities entered in the third, fourth, and fifth columns of the Tables given above be designated a, b, c, then we have these three equations at each comparison

$$Y_{55} + z = Y_{55} + x + fy + a$$
$$Y_{55} + z = Y_{55} + b$$
$$Y_{55} + z = Y_{55} + x' + fy' + c$$

So there results a series of equations as follows

$$
\begin{aligned}
x + f_1 y - z_1 + a_1 &= 0 \\
- z_1 + b_1 &= 0 \\
x' + f_1 y' - z_1 + c_1 &= 0 \\
x + f_2 y - z_2 + a_2 &= 0 \\
- z_2 + b_2 &= 0 \\
x' + f_2 y' - z_2 + c_2 &= 0 \\
x + f_3 y - z_3 + a_3 &= 0 \\
- z_3 + b_3 &= 0 \\
x' + f_3 y' - z_3 + c_3 &= 0
\end{aligned} \tag{3}
$$

and so on. Let n be the number of comparisons, then from these equations we obtain

$$
\begin{aligned}
n x + (f) y - (z) + (a) &= 0 \\
(f) x + (f^2) y - (f z) + (f a) &= 0 \\
n x' + (f) y' - (z) + (c) &= 0 \\
(f) x' + (f^2) y' - (f z) + (f c) &= 0 \\
x + x' + f_1 y + f_1 y' - 3 z_1 + a_1 + b_1 + c_1 &= 0 \\
x + x' + f_2 y + f_2 y' - 3 z_2 + a_2 + b_2 + c_2 &= 0 \\
x + x' + f_3 y + f_3 y' - 3 z_3 + a_3 + b_3 + c_3 &= 0 \\
&\vdots \\
x + x' + f_n y + f_n y' - 3 z_n + a_n + b_n + c_n &= 0
\end{aligned} \tag{4}
$$

Eliminating the z's from the first four equations

$$
\begin{aligned}
\tfrac{2}{3} n x + \tfrac{2}{3} (f) y - \tfrac{1}{3} n x' - \tfrac{1}{3} (f) y' + \tfrac{2}{3} (a) - \tfrac{1}{3} (b) - \tfrac{1}{3} (c) &= 0 \\
\tfrac{2}{3} (f) x + \tfrac{2}{3} (f^2) y - \tfrac{1}{3} (f) x' - \tfrac{1}{3} (f^2) y' + \tfrac{2}{3} (a f) - \tfrac{1}{3} (b f) - \tfrac{1}{3} (c f) &= 0 \\
- \tfrac{1}{3} n x - \tfrac{1}{3} (f) y + \tfrac{2}{3} n x' + \tfrac{2}{3} (f) y' - \tfrac{1}{3} (a) - \tfrac{1}{3} (b) + \tfrac{2}{3} (c) &= 0 \\
- \tfrac{1}{3} (f) x - \tfrac{1}{3} (f^2) y + \tfrac{2}{3} (f) x' + \tfrac{2}{3} (f^2) y' - \tfrac{1}{3} (a f) - \tfrac{1}{3} (b f) + \tfrac{2}{3} (c f) &= 0
\end{aligned} \tag{5}
$$

which contain $x\ x'\ y\ y'$ only. Let P Q R S be the absolute terms of these equations, then

$$
\begin{aligned}
n x + (f) y + 2 P + R &= 0 \\
(f) x + (f^2) y + 2 Q + S &= 0 \\
n x' + (f) y' + P + 2 R &= 0 \\
(f) x' + (f^2) y' + Q + 2 S &= 0
\end{aligned} \tag{6}
$$

And finally

$$
\begin{aligned}
\left(n (f^2) - (f)^2\right) x + 2 (f^2) P - 2 (f) Q + (f^2) R - (f) S &= 0 \\
\left(n (f^2) - (f)^2\right) y - 2 (f) P + 2 n Q - (f) R + n S &= 0 \\
\left(n (f^2) - (f)^2\right) x' + (f^2) P - (f) Q + 2 (f^2) R - 2 (f) S &= 0 \\
\left(n (f^2) - (f)^2\right) y' - (f) P + n Q - 2 (f) R + 2 n S &= 0
\end{aligned} \tag{7}
$$

The reciprocals of the weights of the determinations $x\ y\ x'\ y'$ are, therefore,

$$x \ldots \frac{2\,(f^2)}{n\,(f^2) - (f)^2} \qquad (8)$$

$$y \ldots \frac{2\,n}{n\,(f^2) - (f)^2}$$

$$x' \ldots \frac{2\,(f^2)}{n\,(f^2) - (f)^2}$$

$$y' \ldots \frac{2\,n}{n\,(f^2) - (f)^2}$$

We have also

$$\begin{aligned} 2\,P + R &= (a) - (b) \\ 2\,Q + S &= (af) - (bf) \\ P + 2\,R &= -\,(b) + (c) \\ Q + 2\,S &= -\,(bf) + (cf) \end{aligned} \qquad (9)$$

4.

Let us now apply these equations to the comparisons of the Yards. The following Tables contain in the four first columns the data, viz., f, a, b, c, the latter being reduced to the unit for small quantities, viz., the millionth of a yard; columns five, six, and seven contain the multiples $af\ bf\ cf$ for each comparison.

Comparisons of Standard Yards Y_{00} Y_{55} Y_{57}

$t-62° = f$	a	b	c	$a \cdot f$	$b \cdot f$	$c \cdot f$
−23·74	20·25	28·96	20·58	−480·74	−687·51	−488·57
−23·66	19·37	27·03	18·61	−458·29	−639·53	−440·31
−23·57	18·63	26·28	18·44	−439·11	−619·42	−434·63
−23·62	19·02	26·66	19·07	−449·25	−629·71	−450·43
−23·46	19·97	26·69	19·96	−468·50	−626·15	−468·26
−23·37	19·02	26·57	19·41	−444·50	−620·94	−453·61
+ 0·59	33·35	34·75	34·36	+ 19·68	+ 20·50	+ 20·27
+ 0·71	31·69	31·24	32·58	+ 22·50	+ 22·18	+ 23·13
+ 0·89	30·93	30·98	32·06	+ 27·53	+ 27·57	+ 28·53
+ 1·04	30·28	30·65	31·98	+ 31·49	+ 31·88	+ 33·26
+ 0·90	30·49	31·74	33·00	+ 27·44	+ 28·57	+ 29·70
+ 0·99	31·11	31·13	31·83	+ 30·80	+ 30·82	+ 31·51
+ 0·77	33·19	32·79	33·33	+ 25·56	+ 25·25	+ 25·66
+ 1·17	31·10	31·38	31·34	+ 36·39	+ 36·71	+ 36·67
+ 1·39	30·75	29·67	31·56	+ 42·74	+ 41·24	+ 43·87
+ 0·41	33·48	33·00	34·83	+ 13·73	+ 13·53	+ 14·28
+ 0·49	45·58	47·02	45·44	+ 22·33	+ 23·04	+ 22·27
+ 0·82	44·81	44·92	44·76	+ 36·74	+ 36·83	+ 36·70
+ 0·13	47·01	47·07	47·10	+ 6·11	+ 6·12	+ 6·12
+ 0·29	46·07	46·82	47·20	+ 13·36	+ 13·58	+ 13·69
− 0·18	48·91	48·87	48·99	− 8·80	− 8·80	− 8·82
+ 0·07	46·77	47·10	46·92	+ 3·27	+ 3·30	+ 3·28
+ 0·39	44·66	45·53	46·16	+ 17·42	+ 17·76	+ 18·00
− 0·58	49·63	49·78	50·10	− 28·79	− 28·87	− 29·06
+ 4·02	55·20	54·04	55·07	+221·90	+217·24	+221·38
+ 3·63	55·63	55·34	55·17	+201·94	+200·88	+200·27

COMPARISON OF STANDARD YARDS Y_{65} Y_{55} Y_{20}

$t - 62° = f$	a	b	c	$a \cdot f$	$b \cdot f$	$c \cdot f$
−22·79	14·66	23·15	100·26	−334·10	−527·59	−2284·93
−22·48	14·48	21·81	98·28	−325·51	−490·29	−2209·33
−22·27	12·65	20·54	96·31	−281·72	−457·43	−2144·82
−21·98	12·35	19·80	93·25	−271·45	−435·20	−2049·64
−21·77	11·01	18·35	93·05	−239·69	−399·48	−2025·70
+ 0·50	45·48	44·23	40·99	+ 22·74	+ 22·12	+ 20·50
− 0·03	48·60	47·18	45·47	− 1·46	− 1·42	− 1·36
+ 0·71	46·10	44·82	40·36	+ 32·73	+ 31·82	+ 28·66
+ 1·10	44·40	42·97	37·03	+ 48·84	+ 47·27	+ 40·73
+ 1·78	42·35	40·48	32·96	+ 75·38	+ 72·05	+ 58·67
+ 2·20	40·96	39·18	30·26	+ 90·11	+ 86·20	+ 66·57
+ 2·54	38·42	38·27	27·43	+ 97·59	+ 97·21	+ 69·67
+ 2·82	38·86	36·43	24·28	+109·59	+102·73	+ 68·47
+ 2·91	37·34	35·94	24·45	+108·66	+104·59	+ 71·15
+ 3·19	37·30	36·36	22·73	+118·99	+115·99	+ 72·51
+ 3·77	56·05	55·67	39·70	+211·31	+209·88	+ 149·67
+ 4·05	54·12	54·52	37·89	+219·19	+220·81	+ 153·45
+ 4·26	53·48	52·90	37·12	+227·82	+225·35	+ 158·13
+ 4·53	52·45	51·31	32·96	+237·60	+232·43	+ 149·31
+ 3·32	57·34	58·23	43·20	+190·37	+193·32	+ 143·42
+ 3·74	55·95	54·57	38·06	+209·25	+204·09	+ 142·34
+ 3·97	55·25	54·63	38·26	+219·34	+216·88	+ 151·89
+ 4·21	53·53	52·99	35·44	+225·36	+223·09	+ 149·20
+ 3·78	55·08	55·28	39·60	+208·20	+208·96	+ 149·69
+ 3·97	54·24	53·38	37·93	+215·33	+211·92	+ 150·58

We shall consider first the yards $\mathbf{Y}_{66}$ $\mathbf{Y}_{55}$ $\mathbf{Y}_{67}$. Here we have

$$
\begin{aligned}
(f) &= -\ 123\cdot48 \\
(a) &= \ 916\cdot90 \\
(b) &= \ 966\cdot01 \\
(c) &= \ 929\cdot85 \\
(af) &= -\ 1977\cdot05 \\
(bf) &= -\ 3063\cdot93 \\
(cf) &= -\ 1965\cdot10 \\
(f^2) &= \ 3372\cdot816 \\
n &= \ 26
\end{aligned}
$$

and for the determination of $x\ y\ x'\ y'$ the following are the equations—

$$
\begin{aligned}
26\cdot00\,x - 123\cdot480\,y - 49\cdot11 &= 0 \\
-\,123\cdot48\,x + 3372\cdot816\,y + 1086\cdot88 &= 0 \\
26\cdot00\,x' - 123\cdot480\,y' - 36\cdot16 &= 0 \\
-\,123\cdot48\,x' + 3372\cdot816\,y' + 1098\cdot83 &= 0
\end{aligned}
\qquad (10)
$$

from which are obtained

$$
\begin{aligned}
x &= +\ 0\cdot434 \\
y &= -\ 0\cdot3064 \\
x' &= -\ 0\cdot189 \\
y' &= -\ 0\cdot3327
\end{aligned}
\qquad (11)
$$

and the reciprocals of the weights,

for x or x' . . . . . $\cdot093118$ (12)

for y or y' . . . . . $\cdot000718$

We have then finally for the lengths of $\mathbf{Y}_{66}$ and $\mathbf{Y}_{67}$

$$
\begin{aligned}
\mathbf{Y}_{66} &= \mathbf{Y}_{55} + 0\cdot43 - 0\cdot3064\,(t - 62) \\
\mathbf{Y}_{67} &= \mathbf{Y}_{55} - 0\cdot19 - 0\cdot3327\,(t - 62)
\end{aligned}
\qquad (13)
$$

Again, for the yards $\mathbf{Y}_{65}$ $\mathbf{Y}_{55}$ $\mathbf{Y}_{20}$ we have

$$
\begin{aligned}
(f) &= -\ 53\cdot97 \\
(a) &= 1032\cdot45 \\
(b) &= 1052\cdot99 \\
(c) &= 1187\cdot27 \\
(af) &= 1414\cdot47 \\
(bf) &= 515\cdot30 \\
(cf) &= -\ 8721\cdot17 \\
(f^2) &= 2978\cdot59 \\
n &= 25
\end{aligned}
$$

and for the determination of $x\ y\ x'\ y'$, the following are the equations—

$$
\begin{aligned}
25\cdot00\,x - 53\cdot970\,y - 20\cdot54 &= 0 \\
-\ 53\cdot97\,x + 2678\cdot590\,y + 899\cdot17 &= 0 \\
25\cdot00\,x' - 53\cdot970\,y' + 134\cdot28 &= 0 \\
-\ 53\cdot97\,x' + 2678\cdot590\,y' - 9236\cdot47 &= 0
\end{aligned}
\qquad (14)
$$

from whence there result for $x\ y\ x'\ y'$, the following values

$$
\begin{aligned}
x &= +\ 0\cdot101 \\
y &= -\ 0\cdot3336 \\
x' &= +\ 2\cdot167 \\
y' &= +\ 3\cdot4919
\end{aligned}
\qquad (15)
$$

and the reciprocals of the weights,

for x or x' . . . . . $\cdot083635$ (16)

for y or y' . . . . . $\cdot000782$

Thus we get for the lengths of the yards $\mathbf{Y}_{65}$ and $\mathbf{Y}_{20}$ the following values

$$
\begin{aligned}
\mathbf{Y}_{65} &= \mathbf{Y}_{55} + 0\cdot10 - 0\cdot3336\,(t - 62) \\
\mathbf{Y}_{20} &= \mathbf{Y}_{55} + 2\cdot17 + 3\cdot4919\,(t - 62)
\end{aligned}
\qquad (17)
$$

5.

We shall next inquire into the probable error of these results. For this purpose, the quantities z have been computed, and together with the errors of the different equations are exhibited in the following Table:—

Errors of Comparisons of $Y_{66}\ Y_{55}\ Y_{67}$: $Y_{65}\ Y_{55}\ Y_{29}$

Date.	z	Y_{66}	Y_{55}	Y_{67}	z	Y_{65}	Y_{55}	Y_{29}	Date.
1864:									1864:
Feb. 26	28·40	−0·45	+0·56	−0·11	22·79	−0·43	+0·36	+0·06	Feb. 29
„ „	26·79	+0·26	+0·24	−0·50	21·95	+0·13	−0·14	−0·00	„ „
„ „	26·22	+0·06	+0·06	−0·13	20·48	−0·30	+0·06	+0·24	„ „
„ 27	26·70	−0·01	−0·04	+0·04	19·42	+0·36	+0·38	−0·75	Mar. 1
„ „	27·29	+0·30	−0·60	+0·29	18·64	−0·27	−0·29	+0·56	„ „
„ „	26·73	−0·12	−0·16	+0·27	44·85	+0·56	−0·62	+0·06	July 15
July 8	34·11	−0·51	+0·64	−0·14	47·81	+0·90	−0·63	−0·27	„ 16
„ „	31·76	+0·14	−0·52	+0·39	45·26	+0·70	−0·44	−0·25	„ „
„ „	31·21	−0·12	−0·23	+0·36	43·38	+0·75	−0·41	−0·34	„ „
„ „	30·83	−0·44	−0·18	+0·61	41·23	+0·63	−0·75	+0·12	„ 18
„ 9	31·63	−0·99	+0·11	+0·88	39·87	+0·46	−0·69	+0·24	„ „
„ „	31·23	+0·01	−0·10	+0·08	38·14	−0·47	+0·13	+0·33	„ „
„ 11	33·02	+0·36	−0·23	−0·14	36·92	+1·10	−0·49	−0·62	„ „
„ „	31·10	+0·07	+0·28	−0·34	36·40	+0·07	−0·46	+0·38	„ 19
„ „	30·44	+0·31	−0·77	+0·47	36·25	+0·09	+0·11	−0·21	„ „
„ 12	33·76	+0·02	−0·76	+0·74	55·20	−0·31	+0·47	−0·17	„ 20
„ „	45·99	−0·13	+1·03	−0·90	53·86	−0·99	+0·66	+0·34	„ „
„ „	44·74	+0·25	+0·18	−0·44	53·08	−0·92	−0·18	+1·09	„ „
„ 13	47·11	+0·29	−0·04	−0·24	51·10	−0·06	+0·21	−0·15	„ „
„ „	46·71	−0·30	+0·11	+0·20	57·17	−0·84	+1·06	−0·21	„ 21
„ 14	49·04	+0·36	−0·17	−0·18	54·22	+0·58	+0·35	−0·93	„ „
„ „	47·00	+0·18	+0·10	−0·29	54·32	−0·29	+0·31	−0·03	„ „
„ „	45·45	−0·48	+0·08	+0·39	52·51	−0·28	+0·48	−0·20	„ „
„ 15	50·04	+0·20	−0·26	+0·06	54·72	−0·80	+0·56	+0·25	„ 22
„ 22	54·00	+0·40	+0·04	−0·45	53·45	−0·43	−0·07	+0·51	
„ 23	54·69	+0·26	+0·65	−0·92					

First, for the bars 66, 55, 67, the sum of the squares of the seventy-eight errors is 12·7036, and the equations contain 26 + 4 = 30 unknown quantities, hence the probable error of an equation—

$$= \pm\, 0\cdot 674 \sqrt{\frac{12\cdot 7036}{78 - 30}} = \pm\, 0\cdot 347 \qquad (18)$$

and we have the weight of the determination of x or x' and of y or y' in equation (12), consequently we get for the probable errors of the determinations of

$$x \text{ or } x' \; \ldots\ldots \; \pm\, 0\cdot 347 \sqrt{\cdot 093118} = \pm\, 0\cdot 106$$

$$y \text{ or } y' \; \ldots\ldots \; \pm\, 0\cdot 347 \sqrt{\cdot 000718} = \pm\, 0\cdot 0093$$

Again, for the bars 65, 55, 29, the sum of the squares of the seventy-five errors is 18·9194, and the number of unknown quantities is 25 + 4 = 29, consequently the probable error of an equation is

$$= \pm\, 0\cdot 674 \sqrt{\frac{18\cdot 9194}{75 - 29}} = \pm\, 0\cdot 432 \qquad (19)$$

and the reciprocals of the weights of the determinations of $x\ x'\ y\ y'$ are given in equation (16). Consequently we have the probable errors of

$$x \text{ or } x' \ldots\ldots \pm 0\cdot432\sqrt{\cdot083635} = \pm 0\cdot125$$
$$y \text{ or } y' \ldots\ldots \pm 0\cdot432\sqrt{\cdot000781} = \pm 0\cdot0121$$

6.

The definite results of these comparisons are then as follows:—

First, as to relative lengths in comparison with yard No. 55 at the temperature of 62° Fahrenheit—

$$\begin{aligned} \mathbf{Y}_{20} &= \mathbf{Y}_{55} + 2\cdot17 \pm 0\cdot12 \\ \mathbf{Y}_{65} &= \mathbf{Y}_{55} + 0\cdot10 \pm 0\cdot12 \\ \mathbf{Y}_{66} &= \mathbf{Y}_{55} + 0\cdot43 \pm 0\cdot11 \\ \mathbf{Y}_{67} &= \mathbf{Y}_{53} - 0\cdot19 \pm 0\cdot11 \end{aligned} \quad (20)$$

Again, for the rates of expansion for 1° Fahrenheit—

$$\begin{aligned} \text{Exp. of } \mathbf{Y}_{29} &= \text{Exp. of } \mathbf{Y}_{55} + 3\cdot4919 \pm 0\cdot0121 \\ \text{,,} \quad \mathbf{Y}_{65} &= \quad \text{,,} \quad \mathbf{Y}_{55} - 0\cdot3336 \pm 0\cdot0121 \\ \text{,,} \quad \mathbf{Y}_{66} &= \quad \text{,,} \quad \mathbf{Y}_{55} - 0\cdot3064 \pm 0\cdot0093 \\ \text{,,} \quad \mathbf{Y}_{67} &= \quad \text{,,} \quad \mathbf{Y}_{55} - 0\cdot3327 \pm 0\cdot0093 \end{aligned} \quad (21)$$

On comparing the lengths of these five bars as they now stand with reference to one another with the lengths assigned by Mr. Sheepshanks, we find differences considerably greater than the probable errors of observation would lead one to expect. If $\mathfrak{Y}$ represent the length of the yard in abstract idea, Mr. Sheepshanks' results were—

$$\begin{aligned} \mathbf{Y}_{29} &= \mathfrak{Y} + 4\cdot58 \\ \mathbf{Y}_{55} &= \mathfrak{Y} + 0\cdot43 \\ \mathbf{Y}_{65} &= \mathfrak{Y} - 1\cdot28 \\ \mathbf{Y}_{66} &= \mathfrak{Y} - 0\cdot65 \\ \mathbf{Y}_{67} &= \mathfrak{Y} - 2\cdot69 \end{aligned} \quad (22)$$

The mean length of these five yards was therefore

$$\mathfrak{Y} + 0\cdot08$$

Here we have between $\mathbf{Y}_{29}$ and $\mathbf{Y}_{67}$ a difference of $7\cdot27$, whereas the difference now appears to be $2\cdot36$: between $\mathbf{Y}_{55}$ and $\mathbf{Y}_{65}$ the difference was $1\cdot71$, it now is $0\cdot10$ Again, between $\mathbf{Y}_{55}$ and $\mathbf{Y}_{67}$ the difference was $3\cdot12$, and now is $0\cdot19$. These apparent changes of length are greater than might be explained by errors of observation. With reference to the Bronze bar 29, it is to be remarked that a particle of mercury has at some time got into one of the wells carrying the divided surface and in the attempt to dislodge it, the surface has been scratched, but the line is not materially hurt, and at the point proper for bisection is tolerably good. This might possibly account for a difference of $1\cdot00$ at the very outside. The bars $\mathbf{Y}_{65}$ $\mathbf{Y}_{67}$ as has been stated, were not very elaborately observed by Mr. Sheepshanks.

In order to obtain from the five bars as they now stand, the value of $\mathfrak{Y}$, the most reasonable method is to assume that the mean length of the five bars is the same now as in 1853, that is

$$\mathfrak{Y} + 0\cdot08$$

From the results we have just obtained, equations (20), the mean length at present is

$$\mathbf{Y}_{55} + 0 \cdot 50$$

Equating these values, $\mathfrak{Y} = \mathbf{Y}_{55} + 0 \cdot 42$, and the absolute length at present of the five bars would be

$$\begin{aligned} \mathbf{Y}_{29} &= \mathfrak{Y} + 1 \cdot 75 \\ \mathbf{Y}_{55} &= \mathfrak{Y} - 0 \cdot 42 \\ \mathbf{Y}_{65} &= \mathfrak{Y} - 0 \cdot 32 \\ \mathbf{Y}_{66} &= \mathfrak{Y} + 0 \cdot 01 \\ \mathbf{Y}_{67} &= \mathfrak{Y} - 0 \cdot 61 \\ \text{Mean} &= \mathfrak{Y} + 0 \cdot 08 \end{aligned} \qquad (23)$$

If we were to reject the bars 29, 65, 67, as less perfectly compared in 1853 with *the* standard than 55 and 66, which were compared on a considerable number of days each, we should obtain on the assumption of the mean length of 55 and 66 remaining constant

$$\begin{aligned} \mathbf{Y}_{55} &= \mathfrak{Y} - 0 \cdot 33 \\ \mathbf{Y}_{60} &= \mathfrak{Y} + 0 \cdot 10 \end{aligned}$$

a result sufficiently close to the former.

If we give double weight to Mr. Sheepshanks' values of $\mathbf{Y}_{55}$ and $\mathbf{Y}_{60}$, then we get

$$\mathbf{Y}_{29} + 2\,\mathbf{Y}_{55} + \mathbf{Y}_{65} + 2\,\mathbf{Y}_{66} + \mathbf{Y}_{67} = 7\,\mathfrak{Y} + 0 \cdot 17$$

but from the present comparisons

$$\mathbf{Y}_{29} + 2\,\mathbf{Y}_{55} + \mathbf{Y}_{65} + 2\,\mathbf{Y}_{66} + 2\,\mathbf{Y}_{67} = 7\,\mathbf{Y}_{55} + 2 \cdot 94$$

consequently, equating these values

$$\mathfrak{Y} = \mathbf{Y}_{55} + 0 \cdot 40$$

and the present lengths of the yards stand thus

$$\begin{aligned} \mathbf{Y}_{29} &= \mathfrak{Y} + 1 \cdot 77 \\ \mathbf{Y}_{55} &= \mathfrak{Y} - 0 \cdot 40 \\ \mathbf{Y}_{65} &= \mathfrak{Y} - 0 \cdot 30 \\ \mathbf{Y}_{66} &= \mathfrak{Y} + 0 \cdot 03 \\ \mathbf{Y}_{67} &= \mathfrak{Y} - 0 \cdot 59 \end{aligned} \qquad (24)$$

the bars standing at the temperature of 62° Fahrenheit.

XIV.

DETERMINATION OF THE LENGTH OF THE ROYAL SOCIETY'S PLATINUM METRE.

The metre whose length is here investigated is the "*Mètre à traits.*" This bar, to which so much interest attaches on account of the observations made to determine its precise length in English inches, first, in 1818 by Captain Kater, and again in 1835 by Mr. Baily, was lent by the Royal Society in 1864 to the Director of the Ordnance Survey, in order that a new determination of its length might be made in terms of the present National Standard Yard.

In the Philosophical Transactions for 1818, page 103, will be found Captain Kater's description of two platinum metres, of which this is one; also on pages 88, 89 of the ninth volume of the Memoirs of the Royal Astronomical Society is Mr. Baily's description of the same bars; one a "*Mètre à traits,*" and the other a "*Mètre à bouts.*"

1.

The Mètre à traits is a flat bar of platinum 41 inches in length, an inch wide, and a quarter of an inch thick. The lines which define the metre are exceedingly fine, and when viewed under the micrometer microscopes are sometimes very difficult to "observe," as they are crossed in every direction by the scratches on the surface, and almost lost among them. On this account it was considered necessary in the present operations to bisect the line not with the *cross* of the micrometer microscopes, but with the *straight transverse wire.*

The bar was mounted on a cradle system of eight rollers, the distance apart of the rollers being

$$\frac{40 \cdot 96}{\sqrt{63}} = 5 \cdot 16 \text{ inches.}$$

40·96 being the precise length of the bar in inches. For comparison, it was laid in the same box with the Ordnance Metre, **OM**, whose length is known in terms of the yard $\mathbf{Y}_{55}$ *see* Section IX.

These two bars lying side by side were compared on August 1st, 2d, 3d, and 4th, at temperatures between 63°·7 and 63°·5: and again on December 26th, 27th, 28th, 29th, at temperatures varying from 35°·5 to 38°·3. The number of comparisons in all is 26. In each comparison, each bar is observed twice, the total number of micrometer readings being 32 disposed thus:

Bars.	Microscopes.	
	H	K
RSM	4 readings	4 readings
OM	4 ,,	4 ,,
OM	4 ,,	4 ,,
RSM	4 ,,	4 ,,

The large number—four—of readings taken at a time, being with the view of eliminating the error of observation due to the indistinctness of the lines on the Platinum Metre. A smaller number of readings of **OM** would have sufficed as the lines on that bar are excellent.

In order to ascertain whether the peculiarity of the lines on the Platinum Metre might give rise to "personal error" in the bisection with the transverse wire of the micrometer, a large number of readings of each line were made, on two successive days, by three observers, Captain Clarke, R.E., Quartermaster Steel, R.E., and Corporal Compton, R.E. The result obtained from these, was the appearance of the following personal errors,

Captain Clarke . . . . $+0 \cdot 27$ division.
Quartermaster Steel . . . . $-0 \cdot 12$,,
Corporal Compton . . . . $-0 \cdot 14$,,

these quantities being for each observer the sum of the personal errors on the two microscopes H and K. But as they are smaller than the probable errors attaching to them, we shall dismiss them from further consideration.

On each evening the bars were both dismounted, and the box containing them reversed end for end, so that the bar which had been next the piers during the day would be furthest from the piers on the following day. Both bars were then made level, the microscopes brought to accurate focus, and their axes adjusted to verticality. Thus all the adjustments being renewed each evening, there can be very little fear of anything like constant error attaching to the result.

The comparisons were made generally at about 9^h A.M., 12^h, 3^h, and 5^h P.M., the rule being to keep them as far separated in point of time as convenient.

The middle points of the lines are intended to be observed, but just at those points it is impossible to observe them: in fact no particular *point* of either line was observed, but the general appearance of the line for a space of about two hundreths of an inch on either side of the centre was considered in making the bisections. The centre of the line was indicated by a mark on a small slip of paper lying across the scale.

2.

In order that the extent of the uncertainty attaching to the faintness of the line may be fully apparent, we shall here give, in the following table, all the individual micrometer readings.

Comparison of Royal Society's Metre (λ traits) with Ordnance Metre.

No. of Comparison and Date.	Thermometer.	RSM		OM		No. of Comparison and Date.	Thermometer.	RSM		OM	
		H	K	H	K			H	K	H	K
1864.	°					1864.	°				
Aug. 1	65·12	27·3	41·6	37·3	44·9	Aug. 2	64·76	36·1	37·2	38·1	40·2
	65·18	29·5	43·1	36·9	45·2		64·74	33·2	37·9	37·1	40·9
		28·3	42·1	37·0	44·6			34·0	37·6	38·0	41·2
		29·0	43·3	37·0	45·3			34·0	37·1	38·1	40·6
1						6					
		27·0	43·2	31·0	50·9			35·0	34·8	37·1	41·1
		25·4	46·0	31·9	51·2			36·7	35·1	37·2	41·2
	65·13	28·1	45·8	31·0	51·2		64·80	37·5	33·2	37·1	41·4
	65·20	28·5	47·0	31·0	50·6		64·76	35·8	35·9	38·1	41·6
Aug. 1	· 65·15	35·4	38·6	30·2	53·3	Aug. 2	64·90	33·6	39·6	37·3	40·0
	65·20	36·3	37·5	30·0	54·3		64·86	35·0	38·0	36·0	40·6
		35·2	38·9	29·3	54·3			33·2	38·9	37·0	40·3
		34·2	38·9	29·0	54·3			33·7	36·2	36·6	39·7
2						7					
		35·3	38·6	40·1	41·7			37·6	36·0	36·6	41·0
		36·9	37·9	41·8	42·2			38·4	37·0	37·1	40·8
	65·20	38·1	40·8	42·8	41·6		64·90	37·0	35·9	36·2	40·6
	65·24	38·0	40·1	41·4	41·4		64·87	35·6	37·0	36·0	40·0
Aug. 1	65·40	37·5	37·2	37·2	41·1	Aug. 3	63·78	50·6	30·5	46·9	40·1
	65·48	35·2	37·0	37·3	41·0		63·75	46·8	35·0	47·1	40·5
		33·9	36·5	37·1	40·9			46·5	31·6	46·3	40·1
		34·0	37·7	37·9	41·1			47·6	35·2	46·6	40·0
3											
		37·5	36·6	37·6	41·0			34·1	48·0	47·5	40·0
		36·2	35·1	38·0	41·0			31·7	43·6	47·1	39·6
	65·41	34·3	39·0	37·3	40·8		63·76	32·1	45·0	46·4	40·9
	65·50	34·0	35·5	37·0	41·2		63·75	30·6	42·0	46·6	40·3
Aug. 1	65·46	33·0	36·9	37·4	40·2	Aug. 3	63·90	30·9	48·5	39·1	48·1
	65·53	33·5	36·0	37·6	40·0		63·90	31·0	50·7	39·1	48·0
		35·7	35·3	38·6	40·0			27·9	48·8	39·5	48·0
		34·6	34·6	38·6	40·0			30·0	48·9	39·1	47·6
4						9					
		32·6	36·0	38·1	40·1			32·1	45·5	38·9	46·9
		33·8	38·8	37·6	39·8			32·9	42·1	39·6	47·5
	65·51	34·9	36·8	37·9	40·1		63·96	32·1	42·2	39·8	47·3
	65·54	34·0	38·3	38·2	39·9		63·94	33·7	45·1	39·4	47·3
Aug. 2	64·57	35·2	35·2	39·6	40·6	Aug. 3	64·20	33·7	39·9	36·4	45·8
	64·55	37·8	31·2	39·5	40·3		64·20	36·9	39·1	35·4	45·4
		36·9	34·1	39·4	39·6			35·8	39·8	36·0	45·2
		38·9	35·6	40·1	40·5			37·1	38·2	35·5	45·5
5						10					
		32·2	38·8	39·5	39·5			43·1	34·0	35·9	45·0
		33·0	38·3	39·1	40·1			41·4	34·0	36·1	44·9
	64·58	34·9	39·2	39·2	40·2		64·24	43·4	36·1	36·0	45·2
	64·56	34·0	37·5	39·7	40·1		64·20	39·3	35·9	36·0	45·2

Comparison of Royal Society's Metre (λ traits) with Ordnance Metre—*continued.*

No. of Comparison and Date.	Thermometer.	RSM H	RSM K	OM H	OM K
1864. Aug. 4	63°·90	49·1	26·0	50·4	32·4
	63·90	51·5	23·2	50·2	32·9
		50·9	25·0	50·9	33·4
		50·0	23·1	50·0	33·5
11					
		46·6	31·6	49·0	33·5
		43·1	31·0	49·0	34·2
	63·91	46·0	31·3	49·4	34·1
	63·90	44·5	30·0	49·3	34·1
Aug. 4	64·08	42·4	29·0	46·0	36·6
	64·05	45·3	31·8	46·3	37·0
		41·8	32·9	46·2	37·5
		43·2	32·4	46·1	37·2
12					
		37·4	36·5	46·3	37·3
		35·9	36·2	46·1	36·8
	64·10	37·0	36·9	45·9	37·0
	64·08	37·8	35·6	46·0	37·3
Dec. 26	38·36	6·7	6·9	30·0	32·0
	38·34	6·4	7·5	29·3	32·4
		6·0	8·0	29·4	32·6
		7·5	8·4	30·9	32·1
13					
		5·4	8·0	29·1	32·6
		5·0	8·4	30·0	32·5
	38·37	8·1	8·1	29·3	32·3
	38·36	8·5	5·1	29·3	32·6
Dec. 26	38·32	7·0	8·5	27·8	37·0
	38·38	6·8	10·0	27·2	37·5
		8·5	8·9	27·4	37·0
		8·0	8·2	27·6	38·0
14					
		7·7	6·0	27·5	37·0
		8·5	6·5	28·0	37·4
	38·32	8·1	7·5	27·4	37·2
	38·37	7·8	7·9	28·4	38·0
Dec. 27	36·13	29·5	17·7	52·5	46·2
	36·14	30·5	17·8	52·8	46·9
		30·0	16·7	53·1	47·0
		31·0	17·1	52·0	46·9
15					
		30·3	15·0	61·1	38·0
		29·5	15·2	61·8	37·3
	36·10	30·7	17·1	61·5	37·6
	36·10	31·3	16·1	61·4	37·1

No. of Comparison and Date.	Thermometer.	RSM H	RSM K	OM H	OM K
1864. Dec. 27	36°·05	19·9	27·9	48·0	48·9
	36·06	19·9	29·5	47·9	48·7
		18·1	28·8	47·1	48·2
		18·6	29·4	47·0	48·5
16					
		20·0	25·6	50·1	46·4
		19·6	29·1	49·4	46·5
	36·05	20·6	30·0	50·1	47·0
	36·06	18·5	28·6	50·2	47·2
Dec. 27	36·07	22·3	26·3	49·0	52·0
	36·07	21·0	27·5	49·2	52·0
		21·0	26·4	49·4	53·0
		23·0	26·9	48·8	52·7
17					
		22·3	26·4	46·5	53·1
		24·0	27·0	47·5	54·7
	36·10	25·0	26·0	47·0	53·2
	36·09	25·6	25·9	48·0	53·1
Dec. 27	36·20	26·0	23·0	45·0	51·4
	36·22	27·1	24·2	44·6	50·9
		23·0	21·9	44·4	51·8
		23·2	22·0	45·0	51·4
18					
		24·0	21·1	47·3	49·1
		24·9	23·0	46·4	50·2
	36·24	23·9	21·4	46·0	49·5
	36·25	23·5	24·3	46·4	50·0
Dec. 28	35·60	59·0	22·5	85·2	47·0
	35·54	60·5	19·5	85·5	47·2
		61·5	19·0	85·0	47·8
		62·5	18·0	85·4	48·0
19					
		61·2	17·8	72·8	57·2
		63·3	15·0	73·7	58·5
	35·60	63·0	15·4	73·9	58·5
	35·54	64·9	15·5	73·3	59·2
Dec. 28	35·46	42·0	38·1	74·9	54·9
	35·49	42·1	36·2	74·4	54·7
		42·2	38·2	74·6	55·6
		44·8	37·0	74·6	55·0
20					
		44·9	38·8	65·2	64·4
		43·8	39·1	65·8	64·4
	35·46	44·9	37·1	65·5	65·1
	35·49	45·1	36·1	65·0	64·7

COMPARISON OF ROYAL SOCIETY'S METRE (λ TRAITS) WITH ORDNANCE METRE—*continued.*

No. of Comparison and Date.	Thermometer.	RSM		OM		No. of Comparison and Date.	Thermometer.	RSM		OM	
		H	K	H	K			H	K	H	K
1864.	°						°				
Dec. 28	35·90	47·5	30·1	61·8	68·1	Dec. 29	37·48	40·9	37·3	70·9	57·0
	35·95	50·0	29·1	63·0	67·7		37·48	41·2	37·3	70·1	57·2
		50·9	28·5	63·7	68·2			40·6	40·2	70·8	57·9
		49·8	30·0	63·4	67·0			42·6	37·2	70·1	57·9
21						24					
		38·2	41·0	65·0	65·7			39·0	39·1	62·0	67·7
		33·9	42·2	63·6	65·0			41·8	37·8	61·9	67·1
	35·92	37·9	39·0	64·9	65·0		37·50	40·5	39·4	61·4	67·1
	36·00	41·0	38·0	63·5	65·8		37·50	39·1	37·2	61·2	67·4
Dec. 28	35·20	47·6	34·7	65·1	61·1	Dec. 29	37·58	41·3	39·0	62·1	67·4
	35·22	47·6	34·4	65·0	61·0		37·59	40·2	37·0	62·0	66·3
		45·2	33·2	64·3	61·5			41·2	38·0	62·9	68·0
		47·1	36·3	65·6	61·6			40·7	37·7	62·5	66·4
22						25					
		38·4	41·4	64·8	61·1			35·7	43·1	62·3	67·3
		37·0	41·0	65·5	61·6			35·0	42·8	64·0	67·0
	35·21	38·1	42·0	·65·0	61·2		37·60	33·2	43·2	63·9	67·9
	35·26	38·0	42·2	65·1	62·1		37·62	35·3	43·7	63·5	66·4
Dec. 29	37·30	48·3	35·2	68·4	61·6	Dec. 29	37·76	32·0	44·3	61·8	66·3
	37·30	47·0	34·3	69·0	61·8		37·80	33·1	43·8	61·3	66·5
		46·2	34·6	68·3	60·0			32·7	44·5	61·9	66·8
		48·4	33·3	69·5	61·3			33·7	45·0	61·5	66·9
23						26					
		48·0	34·9	70·3	59·5			31·8	46·5	58·0	68·1
		47·0	34·7	70·0	59·8			33·5	45·0	58·6	69·2
	37·34	45·0	35·0	70·9	60·2		37·80	32·7	46·0	59·7	69·6
	37·37	47·5	34·2	70·1	60·0		37·80	34·0	44·5	59·9	69·9

The thermometer readings here given are corrected for the errors of the respective thermometers.

If now from the individual readings in each group of four we subtract the mean of the four, we shall get a series of errors which may be considered as errors of *reading:* these errors are given in the following table.

Errors of Reading.

No. of Comparison.	RSM		OM	
	H	K	H	K
1	− 1·23	− 0·92	+ 0·25	− 0·10
	+ 0·97	+ 0·58	− 0·15	+ 0·20
	− 0·23	− 0·42	− 0·05	− 0·40
	+ 0·47	+ 0·78	− 0·05	+ 0·30
	− 0·25	− 2·30	− 0·23	− 0·07
	− 1·85	+ 0·50	+ 0·67	+ 0·23
	+ 0·85	+ 0·30	− 0·23	+ 0·23
	+ 1·25	+ 1·50	− 0·23	− 0·37
2	+ 0·13	+ 0·13	+ 0·58	− 0·75
	+ 1·03	− 0·97	+ 0·38	+ 0·25
	− 0·07	+ 0·43	− 0·32	+ 0·25
	− 1·07	+ 0·43	− 0·62	+ 0·25
	− 1·77	− 0·75	− 1·42	− 0·02
	− 0·17	− 1·45	+ 0·28	+ 0·48
	+ 1·03	+ 1·45	+ 1·28	− 0·12
	+ 0·93	+ 0·75	− 0·12	− 0·32
3	+ 2·35	+ 0·10	− 0·17	+ 0·08
	+ 0·05	− 0·10	− 0·07	− 0·02
	− 1·25	− 0·60	− 0·27	− 0·12
	− 1·15	+ 0·60	+ 0·53	+ 0·08
	+ 2·00	+ 0·05	+ 0·13	− 0·00
	+ 0·70	− 1·45	+ 0·53	− 0·00
	− 1·20	+ 2·45	− 0·17	− 0·20
	− 1·50	− 1·05	− 0·47	+ 0·20
4	− 1·20	+ 1·20	− 0·65	+ 0·15
	− 0·70	+ 0·30	− 0·45	− 0·05
	+ 1·50	− 0·40	+ 0·55	− 0·05
	+ 0·40	− 1·10	+ 0·55	− 0·05
	− 1·22	− 1·47	+ 0·15	+ 0·13
	− 0·02	+ 1·33	− 0·35	− 0·17
	+ 1·08	− 0·67	− 0·05	+ 0·13
	+ 0·18	+ 0·83	+ 0·25	− 0·07
5	− 2·00	+ 1·18	− 0·05	+ 0·35
	+ 0·60	− 2·82	− 0·15	+ 0·05
	− 0·30	+ 0·08	− 0·25	− 0·65
	+ 1·70	+ 1·58	+ 0·45	+ 0·25
	− 1·32	+ 0·35	+ 0·13	− 0·47
	− 0·52	− 0·15	− 0·27	+ 0·13
	+ 1·38	+ 0·75	− 0·17	+ 0·23
	+ 0·48	− 0·95	+ 0·33	+ 0·13
6	+ 1·78	− 0·25	+ 0·28	− 0·52
	− 1·12	+ 0·45	− 0·72	+ 0·18
	− 0·32	+ 0·15	+ 0·18	+ 0·48
	− 0·32	− 0·35	+ 0·28	− 0·12
	− 1·25	+ 0·05	− 0·27	− 0·22
	+ 0·45	+ 0·35	− 0·17	− 0·12
	+ 1·25	− 1·55	− 0·27	+ 0·08
	− 0·45	+ 1·15	+ 0·73	+ 0·28
7	− 0·28	+ 1·42	+ 0·57	− 0·15
	+ 1·12	− 0·18	− 0·73	+ 0·45
	− 0·68	+ 0·72	+ 0·27	+ 0·15
	− 0·18	− 1·98	− 0·13	− 0·45
	+ 0·45	− 0·48	+ 0·12	+ 0·40
	+ 1·25	+ 0·52	+ 0·62	+ 0·20
	− 0·15	− 0·58	− 0·28	− 0·00
	− 1·55	+ 0·52	− 0·48	− 0·60
8	+ 2·72	− 2·58	+ 0·17	− 0·08
	− 1·08	+ 1·92	+ 0·37	+ 0·32
	− 1·38	− 1·48	− 0·43	− 0·08
	− 0·28	+ 2·12	− 0·13	− 0·18
	+ 1·97	+ 3·35	+ 0·60	− 0·20
	− 0·43	− 1·05	+ 0·20	− 0·60
	− 0·03	+ 0·35	− 0·50	+ 0·70
	− 1·53	− 2·65	− 0·30	+ 0·10
9	+ 0·95	− 0·73	− 0·10	+ 0·17
	+ 1·05	+ 1·47	− 0·10	+ 0·07
	− 2·05	− 0·43	+ 0·30	− 0·07
	+ 0·05	− 0·33	− 0·10	− 0·33
	− 0·60	+ 1·77	− 0·53	− 0·35
	+ 0·20	− 1·63	+ 0·17	+ 0·25
	− 0·60	− 1·53	+ 0·37	+ 0·05
	+ 1·00	+ 1·37	− 0·03	+ 0·05
10	− 2·18	+ 0·65	+ 0·57	+ 0·32
	+ 1·02	− 0·15	− 0·43	− 0·08
	− 0·08	+ 0·55	+ 0·17	− 0·28
	+ 1·22	− 1·05	− 0·33	+ 0·02
	+ 1·30	− 1·00	− 0·10	− 0·08
	− 0·40	− 1·00	+ 0·10	− 0·18
	+ 1·60	+ 1·10	− 0·00	+ 0·12
	− 2·50	+ 0·90	− 0·00	+ 0·12

Errors of Reading—*continued.*

No. of Comparison.	RSM H	RSM K	OM H	OM K
11	− 1·28	+ 1·67	+ 0·02	− 0·65
	+ 1·12	− 1·13	− 0·18	− 0·15
	+ 0·52	+ 0·67	+ 0·52	+ 0·35
	− 0·38	− 1·23	− 0·38	+ 0·45
	+ 1·55	+ 0·62	− 0·18	− 0·48
	− 1·95	+ 0·02	− 0·18	+ 0·22
	+ 0·95	+ 0·32	+ 0·22	+ 0·12
	− 0·55	− 0·98	+ 0·12	+ 0·12
12	− 0·78	− 2·53	− 0·15	− 0·48
	+ 2·12	+ 0·27	+ 0·15	− 0·08
	− 1·38	+ 1·37	+ 0·05	+ 0·42
	+ 0·02	+ 0·87	− 0·05	+ 0·12
	+ 0·37	+ 0·20	+ 0·22	+ 0·20
	− 1·13	− 0·10	+ 0·02	− 0·30
	− 0·03	+ 0·60	− 0·18	− 0·10
	+ 0·77	− 0·70	− 0·08	+ 0·20
13	+ 0·05	− 0·80	+ 0·10	− 0·28
	− 0·25	− 0·20	− 0·60	+ 0·12
	− 0·65	+ 0·30	− 0·50	+ 0·32
	+ 0·85	+ 0·70	+ 1·00	− 0·18
	− 1·35	+ 0·60	− 0·33	+ 0·10
	− 1·75	+ 1·00	+ 0·57	+ 0·00
	+ 1·35	+ 0·70	− 0·13	− 0·20
	+ 1·75	− 2·30	− 0·13	+ 0·10
14	− 0·58	− 0·40	+ 0·30	− 0·38
	− 0·78	+ 1·10	− 0·30	+ 0·12
	+ 0·92	+ 0·00	− 0·10	− 0·38
	+ 0·42	− 0·70	+ 0·10	+ 0·62
	− 0·33	− 0·98	− 0·33	− 0·40
	+ 0·47	− 0·48	+ 0·17	+ 0·00
	+ 0·07	+ 0·52	− 0·43	− 0·20
	− 0·23	+ 0·92	+ 0·57	+ 0·60
15	− 0·75	+ 0·37	− 0·10	− 0·55
	+ 0·25	+ 0·47	+ 0·20	+ 0·15
	− 0·25	− 0·63	+ 0·50	+ 0·25
	+ 0·75	− 0·23	− 0·60	+ 0·15
	− 0·15	− 0·85	− 0·35	+ 0·50
	− 0·95	− 0·65	+ 0·35	− 0·20
	+ 0·25	+ 1·25	+ 0·05	+ 0·10
	+ 0·85	+ 0·25	− 0·05	− 0·40
16	+ 0·77	− 1·00	+ 0·50	+ 0·32
	+ 0·77	+ 0·60	+ 0·40	+ 0·12
	− 1·03	− 0·10	− 0·40	− 0·38
	− 0·53	+ 0·50	− 0·50	− 0·08
	+ 0·32	− 2·73	+ 0·15	− 0·38
	− 0·08	+ 0·77	− 0·55	− 0·28
	+ 0·92	+ 1·67	+ 0·15	+ 0·22
	− 1·18	+ 0·27	+ 0·25	+ 0·42
17	+ 0·47	− 0·48	− 0·10	− 0·43
	− 0·83	+ 0·72	+ 0·10	− 0·43
	− 0·83	− 0·38	+ 0·30	+ 0·57
	+ 1·17	+ 0·12	− 0·30	+ 0·27
	− 1·93	+ 0·07	− 0·75	− 0·43
	− 0·23	+ 0·67	+ 0·25	+ 1·17
	+ 0·77	− 0·33	− 0·25	− 0·33
	+ 1·37	− 0·43	+ 0·75	− 0·43
18	+ 1·17	+ 0·22	+ 0·25	+ 0·02
	+ 2·27	+ 1·42	− 0·15	− 0·48
	− 1·83	− 0·88	− 0·35	+ 0·42
	− 1·63	− 0·78	+ 0·25	+ 0·02
	− 0·08	− 1·35	+ 0·77	− 0·60
	+ 0·82	+ 0·55	− 0·13	+ 0·50
	− 0·18	− 1·05	− 0·53	− 0·20
	− 0·58	+ 1·85	− 0·13	+ 0·30
19	− 1·88	+ 2·70	− 0·08	− 0·50
	− 0·38	− 0·25	+ 0·22	− 0·30
	+ 0·62	− 0·75	− 0·28	+ 0·30
	+ 1·62	− 1·75	+ 0·12	+ 0·50
	− 1·90	+ 1·87	− 0·63	− 1·15
	+ 0·20	− 0·93	+ 0·27	+ 0·15
	− 0·10	− 0·53	+ 0·47	+ 0·15
	+ 1·80	− 0·43	− 0·13	+ 0·85
20	− 0·78	+ 0·72	+ 0·27	− 0·15
	− 0·68	− 1·18	− 0·23	− 0·35
	− 0·58	+ 0·82	− 0·03	+ 0·55
	+ 2·02	− 0·38	− 0·03	− 0·05
	+ 0·22	+ 1·02	− 0·18	− 0·25
	− 0·88	+ 1·32	+ 0·42	− 0·25
	+ 0·22	− 0·68	+ 0·12	+ 0·45
	+ 0·42	− 1·68	− 0·38	+ 0·05

Errors of Reading—*continued.*

No. of Comparison.	RSM H	RSM K	OM H	OM K	No. of Comparison.	RSM H	RSM K	OM H	OM K
21	− 2·05	+ 0·67	− 1·18	+ 0·35	24	− 0·43	− 0·70	+ 0·42	− 0·50
	+ 0·45	− 0·33	+ 0·02	− 0·05		− 0·13	− 0·70	− 0·38	− 0·30
	+ 1·35	− 0·93	+ 0·72	+ 0·45		− 0·73	+ 2·20	+ 0·32	+ 0·40
	+ 0·25	+ 0·57	+ 0·42	− 0·75		+ 1·27	− 0·80	− 0·38	+ 0·40
	+ 0·45	+ 0·95	+ 0·75	+ 0·32		− 1·10	+ 0·72	+ 0·37	+ 0·37
	− 3·85	+ 2·15	− 0·65	− 0·38		+ 1·70	− 0·58	+ 0·27	− 0·23
	+ 0·15	− 1·05	+ 0·65	− 0·38		+ 0·40	+ 1·02	− 0·23	− 0·23
	+ 3·25	− 2·05	− 0·75	+ 0·42		− 1·00	− 1·18	− 0·43	+ 0·07
22	+ 0·72	+ 0·05	+ 0·10	− 0·20	25	+ 0·45	+ 1·07	− 0·28	+ 0·37
	+ 0·72	− 0·25	+ 0·00	− 0·30		− 0·65	− 0·93	− 0·38	− 0·73
	− 1·68	− 1·45	− 0·70	+ 0·20		+ 0·35	+ 0·07	+ 0·52	+ 0·97
	+ 0·22	+ 1·65	+ 0·60	+ 0·30		− 0·15	− 0·23	+ 0·12	− 0·63
	+ 0·52	− 0·25	− 0·30	− 0·40		+ 0·90	− 0·10	− 1·13	+ 0·15
	− 0·88	− 0·65	+ 0·40	+ 0·10		+ 0·20	− 0·40	+ 0·57	− 0·15
	+ 0·22	+ 0·35	− 0·10	− 0·30		− 1·60	+ 0·00	+ 0·47	+ 0·75
	+ 0·12	+ 0·55	+ 0·00	+ 0·60		+ 0·50	+ 0·50	+ 0·07	− 0·75
23	+ 0·82	+ 0·87	− 0·40	+ 0·42	26	− 0·88	− 0·10	+ 0·17	− 0·33
	− 0·48	− 0·05	+ 0·20	+ 0·62		+ 0·22	− 0·60	− 0·33	− 0·13
	− 1·28	+ 0·25	− 0·50	− 1·18		− 0·18	+ 0·10	+ 0·27	+ 0·17
	+ 0·92	− 1·05	+ 0·70	+ 0·12		+ 0·82	+ 0·60	− 0·13	+ 0·27
	+ 1·12	+ 0·20	− 0·03	− 0·38		− 1·20	+ 1·00	− 1·05	− 1·10
	+ 0·12	+ 0·00	− 0·33	− 0·08		+ 0·50	− 0·50	− 0·45	− 0·00
	− 1·88	+ 0·30	+ 0·57	+ 0·32		− 0·30	+ 0·50	+ 0·65	+ 0·40
	+ 0·62	− 0·50	− 0·23	+ 0·12		+ 1·00	− 1·00	+ 0·85	+ 0·70

We have here 26 × 16 errors of reading of each scale. The sum of the squares of these errors is

for Royal Society's Metre . . . . 487·38
for Ordnance Metre . . . . 64·24

and from this we deduce the probable error of a single reading

$$\text{for Royal Society's Metre} \ldots \pm \cdot 674\sqrt{\frac{487\cdot 38}{416 - 104}} = \pm\, 0\cdot 842 \qquad (1)$$

$$\text{for Ordnance Metre} \ldots\ldots \pm \cdot 674\sqrt{\frac{64\cdot 24}{416 - 104}} = \pm\, 0\cdot 306 \qquad (2)$$

From this it appears that the probable error in observing a good line—as those on the Ordnance Metre,—is the same when the straight transverse wire is used as when the cross is used; for the probable error in the latter case is (*see* page 63) ± 0·31 and ± 0·32 in the case of two different observers. But the probable error of a bisection of the lines on the Platinum Metre is greater in the proportion of 2·75 : 1·00.

We shall now give in the following Table the mean results of the 26 comparisons.

Date.	Temp.	RSM	OM	Difference in Micrometer Divisions.	RSM − OM
1864.	°				
Aug. 1	65·16	27·89 *h* + 44·01 *k*	31·14 *h* + 47·99 *k*	+ 3·25 *h* + 3·98 *k*	+ 5·76
,, ,,	65·20	36·18 *h* + 38·91 *k*	35·58 *h* + 47·89 *k*	− 0·60 *h* + 8·98 *k*	+ 6·69
,, ,,	65·45	35·33 *h* + 36·83 *k*	37·43 *h* + 41·01 *k*	+ 2·10 *h* + 4·18 *k*	+ 5·00
,, ,,	65·51	34·01 *h* + 36·59 *k*	38·00 *h* + 40·01 *k*	+ 3·99 *h* + 3·42 *k*	+ 5·90
,, 2	64·57	35·36 *h* + 36·24 *k*	39·51 *h* + 40·11 *k*	+ 4·15 *h* + 3·87 *k*	+ 6·39
,, ,,	64·77	35·29 *h* + 36·10 *k*	37·60 *h* + 41·03 *k*	+ 2·31 *h* + 4·93 *k*	+ 5·77
,, ,,	64·88	35·51 *h* + 37·33 *k*	36·60 *h* + 40·38 *k*	+ 1·09 *h* + 3·05 *k*	+ 3·30
,, 3	63·76	40·00 *h* + 38·86 *k*	46·81 *h* + 40·19 *k*	+ 6·81 *h* + 1·33 *k*	+ 6·47
,, ,,	63·93	31·33 *h* + 46·48 *k*	39·31 *h* + 47·59 *k*	+ 7·98 *h* + 1·11 *k*	+ 7·23
,, ,,	64·21	38·84 *h* + 37·13 *k*	35·91 *h* + 45·28 *k*	− 2·93 *h* + 8·15 *k*	+ 4·17
,, 4	63·90	47·71 *h* + 27·65 *k*	49·78 *h* + 33·51 *k*	+ 2·07 *h* + 5·86 *k*	+ 6·32
,, ,,	64·08	40·10 *h* + 33·91 *k*	46·11 *h* + 37·09 *k*	+ 6·01 *h* + 3·18 *k*	+ 7·31
Dec. 26	38·36	6·70 *h* + 7·55 *k*	29·66 *h* + 32·39 *k*	+ 22·96 *h* + 24·84 *k*	+ 38·07
,, ,,	38·35	7·80 *h* + 7·94 *k*	27·66 *k* + 37·39 *k*	+ 19·86 *h* + 29·45 *k*	+ 39·29
,, 27	36·12	30·35 *h* + 16·59 *k*	57·03 *h* + 42·13 *k*	+ 26·68 *h* + 25·54 *k*	+ 41·59
,, ,,	36·06	19·40 *h* + 28·61 *k*	48·73 *h* + 47·68 *k*	+ 29·33 *h* + 19·07 *k*	+ 38·53
,, ,,	36·08	23·03 *h* + 26·55 *k*	48·18 *h* + 52·98 *k*	+ 25·15 *h* + 26·43 *k*	+ 41·08
,, ,,	36·23	24·45 *h* + 22·61 *k*	45·64 *h* + 50·54 *k*	+ 21·19 *h* + 27·93 *k*	+ 39·13
,, 28	35·57	61·99 *h* + 17·84 *k*	79·35 *h* + 52·93 *k*	+ 17·36 *h* + 35·09 *k*	+ 41·80
,, ,,	35·48	43·73 *h* + 37·58 *k*	70·00 *h* + 59·85 *k*	+ 26·27 *h* + 22·27 *k*	+ 38·65
,, ,,	35·94	43·65 *h* + 34·74 *k*	63·61 *h* + 66·56 *k*	+ 19·96 *h* + 31·82 *k*	+ 41·26
,, ,,	35·22	42·38 *h* + 38·15 *k*	65·05 *h* + 61·40 *k*	+ 22·67 *h* + 23·25 *k*	+ 36·57
,, 29	37·33	47·18 *h* + 34·53 *k*	69·56 *h* + 60·53 *k*	+ 22·38 *h* + 26·00 *k*	+ 38·54
,, ,,	37·49	40·71 *h* + 38·19 *k*	66·05 *h* + 62·41 *k*	+ 25·34 *h* + 24·22 *k*	+ 39·47
,, ,,	37·60	37·83 *h* + 40·56 *k*	62·90 *h* + 67·09 *k*	+ 25·07 *h* + 26·53 *k*	+ 41·10
,, ,,	37·79	32·94 *h* + 44·95 *k*	60·34 *h* + 67·91 *k*	+ 27·40 *h* + 22·96 *k*	+ 40·10

3.

Let the excess of length of the Platinum Metre above **OM**, both being at 62°, be x_{62} and when both are at 32° let the difference be x_{32}: then the difference of length at the temperature t

$$x_t = x_{32} \cdot \frac{62 - t}{30} + x_{62} \frac{t - 32}{30}$$

The preceding Table for 26 values of t gives us 26 values of x_t, thus we obtain as many equations of condition between x_{32} and x_{62}. Applying to them the method of least squares, we get

$$14 \cdot 549724\, x_{62} + 0 \cdot 684874\, x_{32} - 163 \cdot 0507 = 0 \qquad (3)$$
$$0 \cdot 684874\, x_{62} + 10 \cdot 080524\, x_{32} - 462 \cdot 4393 = 0$$

If we write A and B for the absolute terms of these equations, they become on eliminating x_{32} and x_{62}

$$x_{62} + \cdot 0689504\, A - \cdot 0046846\, B = 0 \qquad (4)$$
$$x_{32} - \cdot 0046846\, A + \cdot 0995217\, B = 0$$

restoring the values of A and B we get

$$x_{32} = + 45 \cdot 26 \qquad (5)$$
$$x_{62} = + \ \ 9 \cdot 08$$

These values substituted in the equations of condition give the following system of errors corresponding to the 26 comparisons.

No.	Error.	No.	Error.
1	− 0·49	14	− 1·69
2	− 1·47	15	− 1·30
3	− 0·08	16	+ 1·83
4	− 1·05	17	− 0·74
5	− 0·41	18	+ 1·03
6	− 0·03	19	− 0·85
7	+ 2·31	20	+ 2·41
8	+ 0·49	21	− 0·75
9	− 0·48	22	+ 4·81
10	+ 2·24	23	+ 0·29
11	+ 0·47	24	− 0·83
12	− 0·74	25	− 2·59
13	− 0·48	26	− 1·82

The sum of the squares of these errors is 66·03, so that the probable error of a single comparison is

$$\pm 0 \cdot 674 \sqrt{\frac{66 \cdot 03}{24}} = \pm 1 \cdot 118 \qquad (6)$$

and the probable errors of x_{62}, x_{32} are consequently

$$x_{62} \ldots\ldots \pm 1 \cdot 118 \sqrt{\cdot 0690} = \pm 0 \cdot 29 \qquad (7)$$
$$x_{32} \ldots\ldots \pm 1 \cdot 118 \sqrt{\cdot 0995} = \pm 0 \cdot 35$$

4.

It is stated by Captain Kater with reference to the Platinum Metre—"this metre previous to being brought from Paris was compared with a Standard Metre by *M. Arago*, with all that care and ability which he is so well known to possess, and which so delicate an operation requires; the result was that the distance between the lines was found to be less than a metre by $\frac{17 \cdot 59}{1000}$ of a millimetre or ·00069 of an inch": the temperature of the bar being 32°.

We have ascertained that, both bars being at 32°, the Platinum Metre **RSM** is greater than **OM** by $45 \cdot 26 \pm 0 \cdot 35$, and we must now ascertain the length of **OM** at 32° with reference to $\mathbf{Y}_{55}$ at 62°. For this purpose let $[a \cdot b]$, $[b \cdot c]$ represent the yard and the small space on the bar **OM**, so that $[a \cdot c]$ represents its whole length; let the expansion of $\mathbf{Y}_{65}$ be y_{55}, that of a yard of **OM**, $y_{55} + y$. Also at 62° let $[a \cdot b]$ exceed $\mathbf{Y}_{65}$ (also at 62°) by x. Let α represent the fraction $\frac{338}{3600}$, z the excess of $[\mu \cdot e]$ on **OF** above $\frac{3 \cdot 38}{12}$ **F**, x' the excess—at 62°—of **F** above $\frac{1}{3}$ $\mathbf{Y}_{65}$, y' the excess of the expansion of **OF** above that of a foot of $\mathbf{Y}_{55}$, u the excess of $[b \cdot c]$ above α $\mathbf{Y}_{55}$ at 62°, s the excess of $[b \cdot c]$

above $[\mu\cdot e]$ at the temperature $62 + f'$ at which the comparison was made. Then we have the following equations:

$$[\mu\cdot e] = 3\,\alpha\,\mathbf{F} + z$$
$$\mathbf{F} = \tfrac{1}{3}\,\mathbf{Y}_{55} + x' + fy'$$
$$[a\cdot b] = \mathbf{Y}_{55} + x + fy$$
$$[b\cdot c] = \alpha\,\mathbf{Y}_{55} + \alpha fy + u$$
$$[\mu\cdot e] = \alpha\,\mathbf{Y}_{55} + 3\,\alpha\,x' + z + 3\,\alpha fy'$$
$$\therefore\quad s = \alpha f'y + u - 3\,\alpha\,x' - z - 3\,\alpha f'y'$$

and $[a\cdot c] = (1 + \alpha)\,\mathbf{Y}_{55} + (1 + \alpha)fy + x + s - \alpha f'y + 3\,\alpha\,x' + 3\,\alpha f'y' + z$

which last is the expression for the length of **OM** at the temperature t, $\mathbf{Y}_{55}$ being at the same temperature. At 62°, when $f = 0$

$$[a\cdot c] = (1 + \alpha)\,\mathbf{Y}_{55} + s + z + x - \alpha f'y + 3\,\alpha\,(x' + f'y')$$

Now the difference of length of $[a\,c]$, that is **OM**, between the temperatures 32° and 62° is $30\,(1 + \alpha)\,(y_{55} + y)$: hence the length of **OM** at 32° is

$$= (1 + \alpha)\,\mathbf{Y}_{55} + s + z + x - \alpha f'y - 30\,(1 + \alpha)\,y + 3\,\alpha\,(x' + f'y') - 30\,(1 + \alpha)\,y_{55} \qquad (8)$$

When $\mathbf{Y}_{55}$ is at 62° not 32°. The probable error of this expression arises from five different sources:

1. The comparison of $[b\cdot c]$ and $[\mu\cdot e]$
2. The determination of the error of the space $[\mu\cdot e]$ on **OF**
3. The comparisons of $[a\cdot b]$ with $\mathbf{Y}_{55}$
4. The comparisons of **F** with $\mathbf{Y}_{55}$
5. The absolute expansion of $\mathbf{Y}_{55}$

Now from equations (2) and (6), pages 106, 107, the probable error of $x - (30 + 30\,\alpha + \alpha f')\,y$ is; putting $f_{,} = 30 + 30\,\alpha + \alpha f'$

$$\pm\, 0\cdot 412\,(\cdot 03212 - \cdot 00310 f_{,} + \cdot 0003376 f_{,}^{2})^{\frac{1}{2}}$$

Also by equation (8), page 77, the probable error of $x' + f'y'$ is

$$\pm\,(\cdot 01171 + \cdot 00126 f' + \cdot 000049 f'^{\,2})^{\frac{1}{2}}$$

The value of f' is $3\cdot 32$; hence $f_{,} = 33\cdot 13$, and the probable errors specified become $\pm\, 0\cdot 226$ and $\pm\, 0\cdot 128$. This last when multiplied as in the expression for $[a\cdot c]$ by $3\,\alpha$ becomes $\pm\,\cdot 036$. Now, page 108, equation (11), the probable error of s is $\pm\,\cdot 091$; and page 71, equation (47), the probable error of z is $\pm\,\cdot 098$; therefore the probable error of the determination of the length of **OM** at 32° is

$$\pm\left\{(\cdot 091)^2 + (\cdot 098)^2 + (\cdot 226) + (\cdot 036)^2 + (32\cdot 82\,y_{55})^2\right\}^{\frac{1}{2}}$$

if for y_{55} we take the determination at page 90, $y_{55} = \pm\,\cdot 0284$ this becomes

$$\pm\left\{(091)^2 + (\cdot 098)^2 + (\cdot 226)^2 + (\cdot 036)^2 + (\cdot 932)^2\right\}^{\frac{1}{2}} = \pm\, 0\cdot 969 \qquad (9)$$

The actual value of y_{55} just referred to is $6 \cdot 358$, hence $y_{55} + y = 5 \cdot 9471$, and $30 (1 + \alpha) (y_{55} + y) = 195 \cdot 14$. Subtracting this from the length of **OM** at 62° given at equation (22), page 110, we get for the length of **OM** at 32°

$$(1 \cdot 09355830 \pm \cdot 00000097) \mathbf{Y}_{55} \qquad (10)$$

the yard $\mathbf{Y}_{55}$ being at 62°.

The Platinum Metre at 32° exceeds this by $45 \cdot 26 \pm 0 \cdot 35$; hence we have finally for the length of this bar at its standard temperature

$$\text{Royal Society's Metre } (\grave{a} \textit{ traits}) = (1 \cdot 09360356 \pm 0 \cdot 00000103) \mathbf{Y}_{55} \qquad (11)$$

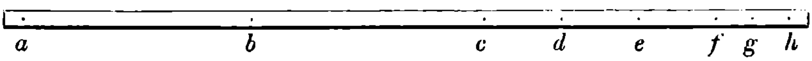
From this result it is to be inferred that the length of the *Standard Metre* referred to by M. Arago is

$$1 \cdot 09362280 \mathbf{Y}_{55} \qquad (12)$$

XV.

AUSTRALIAN 10 FEET STANDARDS.

The two ten feet Iron Bars $\mathbf{OI}_4$ and $\mathbf{OI}_6$ were sent out, the former in December 1858 to Sydney for the Government of New South Wales, and the latter in March 1862 to Melbourne for the Government of Victoria. These bars are of wrought iron and similar in section to the bar **OI**, described page 13. But they differ in the distribution of the points on the upper surface: in the bars $\mathbf{OI}_4$ $\mathbf{OI}_6$ the points are distributed as follows:



$$[a\cdot b] = [b\cdot c] = \text{one yard}$$
$$[c\cdot d] = [d\cdot e] = [e\cdot f] = [f\cdot h] = \text{one foot}$$
$$[f\cdot g] = [g\cdot h] = \text{six inches.}$$

1.

The comparisons of the subdivisions of $\mathbf{OI}_4$ with one another, and with the Copy No. 55 of the Standard Yard, were made by Serjeant Jenkins, R.E., and Serjeant-Major (now Quartermaster) Steel, R.E., in the spring and summer of 1858.

$[a\cdot b]$ was compared with $\mathbf{Y}_{55}$	;	105 comparisons.
$[b\cdot c]$,, ,, $\mathbf{Y}_{55}$	;	145 comparisons.
$[c\cdot f]$,, ,, $\mathbf{Y}_{55}$	;	255 comparisons.
$[c\cdot d]$, $[d\cdot e]$, $[e\cdot f]$, $[f\cdot h]$ together	;	115 comparisons.

Here by a comparison is to be understood a single reading of either extremity of the one measure with the same of the other measure.

The observations were taken in nearly the same numbers by the two observers, and the results are as follows: at 62°00 Fahrenheit,—

	Steel.		Jenkins.
$[a\cdot b] =$	$\mathbf{Y}_{55} + 47\cdot 36$	;	$\mathbf{Y}_{55} + 49\cdot 08$
$[b\cdot c] =$	$\mathbf{Y}_{55} - 51\cdot 86$	;	$\mathbf{Y}_{55} - 49\cdot 83$
$[c\cdot f] =$	$\mathbf{Y}_{55} + 0\cdot 06$	;	$\mathbf{Y}_{55} + 0\cdot 61$
$[c\cdot d] =$	$[f\cdot h] + 89\cdot 57$	;	$[f\cdot h] + 90\cdot 72$
$[d\cdot e] =$	$[f\cdot h] - 35\cdot 13$	;	$[f\cdot h] - 35\cdot 20$
$[e\cdot f] =$	$[f\cdot h] + 21\cdot 12$	;	$[f\cdot h] + 22\cdot 16$

The last three equations are equivalent to the following:

	Steel.		Jenkins.
$[f \cdot h] =$	$\frac{1}{3}[c \cdot f] - 25 \cdot 19$	;	$\frac{1}{3}[c \cdot f] - 25 \cdot 89$
$=$	$\frac{1}{3} \mathbf{Y}_{55} - 25 \cdot 17$	;	$\frac{1}{3} \mathbf{Y}_{55} - 25 \cdot 69$

We have then for the values of the intervals as determined by the two observers the following:

Differences.	Observers.		Mean.
	Steel.	Jenkins.	
$[a \cdot b] - \mathbf{Y}_{55}$	+ 47·36	+ 49·08	+ 48·22
$[b \cdot c] - \mathbf{Y}_{55}$	− 51·86	− 49·83	− 50·85
$[c \cdot d] - \frac{1}{3} \mathbf{Y}_{55}$	+ 64·40	+ 65·03	+ 64·72
$[d \cdot e] - \frac{1}{3} \mathbf{Y}_{55}$	− 60·30	− 60·89	− 60·60
$[e \cdot f] - \frac{1}{3} \mathbf{Y}_{55}$	− 4·05	− 3·53	− 3·79
$[f \cdot h] - \frac{1}{3} \mathbf{Y}_{55}$	− 25·17	− 25·69	− 25·43
$[a \cdot h] - \frac{10}{3} \mathbf{Y}_{55}$	− 29·62	− 25·83	− 27·73

The last line in this table being the sum of the other six gives for the length of the bar at 62°:

$$\mathbf{O\,I}_4 = 3{\cdot}33330560\ \mathbf{Y}_{55}$$

and for the subdivisions the following values:

$$[a \cdot b] = 1 \cdot 00004822\ \mathbf{Y}_{55}$$
$$[b \cdot c] = 0 \cdot 99994915\ \mathbf{Y}_{55}$$
$$[c \cdot d] = 0 \cdot 33339805\ \mathbf{Y}_{55}$$
$$[d \cdot e] = 0 \cdot 33327273\ \mathbf{Y}_{55}$$
$$[e \cdot f] = 0 \cdot 33332954\ \mathbf{Y}_{55}$$
$$[f \cdot h] = 0 \cdot 33330790\ \mathbf{Y}_{55}$$

2.

The observations for the determination of the length of $\mathbf{O\,I}_6$ were made in 1860 by Serjeant-Major Steel, R.E.:

$[a \cdot b]$ was compared with	$\mathbf{Y}_{55}$	;	45 comparisons.
$[b \cdot c]$,, ,,	$\mathbf{Y}_{55}$	;	42 comparisons.
$[c \cdot f]$,, ,,	$\mathbf{Y}_{55}$	;	78 comparisons.
$[d \cdot h]$,, ,,	$\mathbf{Y}_{55}$	;	99 comparisons.
$[c \cdot d]$, $[d \cdot e]$, $[e \cdot f]$, $[f \cdot h]$, together		;	73 comparisons.

Where by a comparison is to be understood a single reading of either extremity of the one measure with the same of the other.

The results of the comparisons are as follows, at 62°·00

$$[a \cdot b] - \mathbf{Y}_{55} = +20 \cdot 88$$
$$[b \cdot c] - \mathbf{Y}_{55} = -16 \cdot 27$$
$$[c \cdot f] - \mathbf{Y}_{55} = +54 \cdot 34$$
$$[d \cdot h] - \mathbf{Y}_{55} = +3 \cdot 19$$
$$[c \cdot d] - [f \cdot h] = +53 \cdot 95$$
$$[d \cdot e] - [f \cdot h] = +93 \cdot 49$$
$$[e \cdot f] - [f \cdot h] = +64 \cdot 80$$

Here we have more than sufficient data, and we must proceed by the method of least squares to find the values of the four foot-intervals. Let $[c \cdot d]$, $[d \cdot e]$, $[e \cdot f]$, $[f \cdot h]$, be equal respectively to $\frac{1}{3}\mathbf{Y}_{55} + x$, $\frac{1}{3}\mathbf{Y}_{55} + y$, $\frac{1}{3}\mathbf{Y}_{55} + z$, $\frac{1}{3}\mathbf{Y}_{55} + w$; then our last five equations become,

$$x + y + z - 51 \cdot 25 = 0$$
$$y + z + w - 3 \cdot 19 = 0$$
$$x - w - 53 \cdot 95 = 0$$
$$y - w - 93 \cdot 49 = 0$$
$$z - w - 64 \cdot 80 = 0$$

From these we get

$$2x + y + z - w - 105 \cdot 20 = 0$$
$$x + 3y + 2z - 147 \cdot 93 = 0$$
$$x + 2y + 3z - 119 \cdot 24 = 0$$
$$-x + 4w + 209 \cdot 05 = 0$$

whence

$$x = -0 \cdot 37$$
$$y = +41 \cdot 14$$
$$z = +12 \cdot 45$$
$$w = -52 \cdot 35$$

We have therefore the following values for the intervals:

$$[a \cdot b] = 1 \cdot 00002088\ \mathbf{Y}_{55}$$
$$[b \cdot c] = 0 \cdot 99998373\ \mathbf{Y}_{55}$$
$$[c \cdot d] = 0 \cdot 33333296\ \mathbf{Y}_{55}$$
$$[d \cdot e] = 0 \cdot 33337447\ \mathbf{Y}_{55}$$
$$[e \cdot f] = 0 \cdot 33334578\ \mathbf{Y}_{55}$$
$$[f \cdot h] = 0 \cdot 33328098\ \mathbf{Y}_{55}$$

and the sum of these gives the length of the Bar at 62°·00

$$\mathbf{O}\,\mathrm{I}_0 = 3{\cdot}33333880\ \mathbf{Y}_{55}$$

3.

Each of the bars $\mathbf{O\,I}_4$ and $\mathbf{O\,I}_6$ was compared with the Ordnance Standard $\mathbf{O}_1$ with the following results:

The Bar $\mathbf{O\,I}_4$ was compared 66 times with $\mathbf{O}_1$ at the mean temperature of 63°·90 Fahr.; and 17 times at the mean temperature of 36°·70, giving

$$\mathbf{O}_1 = \mathbf{O\,I}_4 + 30{\cdot}56 \quad ; \quad \text{at } 36^\circ{\cdot}70$$
$$\mathbf{O}_1 = \mathbf{O\,I}_4 + 28{\cdot}90 \quad ; \quad \text{,, } 63^\circ{\cdot}90$$

These two series of observations were by the same observer, Serjeant-Major Steel, R.E. From them it appears that the rate of expansion of $\mathbf{O\,I}_4$ is very slightly greater than that of $\mathbf{O}_1$; or for each 1° Fahrenheit

$$\text{Expansion of } \mathbf{O\,I}_4 = \text{Expansion of } \mathbf{O}_1 + 0{\cdot}061$$

From this we find

$$\mathbf{O}_1 = \mathbf{O\,I}_4 + 29{\cdot}02 \quad ; \text{ at } 62^\circ{\cdot}00$$

Besides these, we have 50 comparisons at the mean temperature of 64°·94 by Serjeant Jenkins, R.E., which give

$$\mathbf{O}_1 = \mathbf{O\,I}_4 + 28{\cdot}43 \quad ; \text{ at } 64^\circ{\cdot}94$$

which reduced according to the rate of expansion given above is

$$\mathbf{O}_1 = \mathbf{O\,I}_4 + 28{\cdot}61 \quad ; \text{ at } 62^\circ{\cdot}00$$

This agrees very closely with the former determination. The mean of the two gives finally at 62°·00 Fahr.

$$\mathbf{O}_1 = \mathbf{O\,I}_4 + 28{\cdot}81$$

We are now able to infer the ratio of $\mathbf{O}_1$ and $\mathbf{O\,I}_4$ for we have found

$$\mathbf{O\,I}_4 = \tfrac{10}{3}\mathbf{Y}_{55} - 27{\cdot}73$$
$$\mathbf{O}_1 = \mathbf{O\,I}_4 + 28{\cdot}81$$

and the sum of these equations gives

$$\mathbf{O}_1 = \tfrac{10}{3}\mathbf{Y}_{55} + 1{\cdot}08 \ldots\ldots\ldots\ldots (\alpha)$$

The bars $\mathbf{O\,I}_6$ and $\mathbf{O}_1$ were compared together by Serjeant-Major Steel, R.E., 67 times at the mean temperature of 58°·90, and 30 times at the mean temperature of 44°·70, with results as follows:

$$\mathbf{O\,I}_6 = \mathbf{O}_1 - 3{\cdot}82 \quad ; \text{ at } 58^\circ{\cdot}90$$
$$\mathbf{O\,I}_6 = \mathbf{O}_1 - 4{\cdot}07 \quad ; \text{ at } 44^\circ{\cdot}70$$

From this it appears that the expansion for 1° Fahr. of $\mathbf{O\,I}_6$ exceeds that of $\mathbf{O}_1$ by

$$\frac{4{\cdot}07 - 3{\cdot}82}{14{\cdot}2} = 0{\cdot}0176$$

and by this rate of expansion the difference in length of $\mathbf{O}_1$ above $\mathbf{O\,I}_6$ at 62°·00 is 3·77.

We have then the two equations

$$\mathbf{OI}_6 = \tfrac{10}{3}\mathbf{Y}_{55} + 5 \cdot 47$$
$$\mathbf{O}_1 = \mathbf{OI}_6 + 3 \cdot 77$$

which give,

$$\mathbf{O}_1 = \tfrac{10}{3}\mathbf{Y}_{55} + 9 \cdot 24 \;\dots\dots\dots\; (\beta)$$

4.

In the preceding paragraph are contained two entirely independent results (α), (β), for the ratio of the Ordnance Standard $\mathbf{O}_1$ and the Copy No. 55 of the Standard Yard. These results differ by 8·16 or ·00000816 yard, which is, perhaps, larger than might have been expected. But the number of steps leading to each is large, and it is to be noted that the diameter of either of the dots on $\mathbf{O}_1$ is about ·000075 yard = ·0027 inch.

The value of $\mathbf{O}_1$ obtained through $\mathbf{OI}_4$ is the result of 620 comparisons by two observers, that obtained through $\mathbf{OI}_6$ is from 337 comparisons by one observer. If we, therefore, give the former result double the weight of the latter, we get finally

$$\mathbf{O}_1 = \tfrac{10}{3}\mathbf{Y}_{55} + \frac{2 \cdot 16 + 9 \cdot 24}{2 + 1}$$

or, $$\mathbf{O}_1 = \tfrac{10}{3}\mathbf{Y}_{55} + 3 \cdot 80 = 3 \cdot 33333713\, \mathbf{Y}_{55}$$

the temperature of both bars being 62°·00.

This determination agrees very satisfactorily with that given at page 94.

XVI.

DETERMINATION OF THE ABSOLUTE EXPANSIONS OF THE INDIAN 10FT. STANDARDS

Is Ib.

1.

The two new Standard Bars for the Trigonometrical Survey of India are the same in section and in the disposition of the points on their upper surface as the Ordnance Intermediate Bar, described page 13. There is, however, no groove along the upper surface, and the small circular prepared surfaces are slightly depressed *below* the general surface of the bar. The lines are drawn on gold pins. One of the bars is of cast steel, hammered, tough, not hard, and remarkably homogeneous. The other is of Baily's metal, an alloy formed in the proportions,—copper 16, tin $2\frac{1}{2}$, zinc 1. In the upper surface of each bar are eight thermometer wells; two, close together, in the centre of the left yard; two, close together, in the centre of the right yard, and two at (half an inch on either side of) the centre of the bar: besides these are two wells at one-fourth and three-fourths of the bar's length.* The first six are for one set of six thermometers, the other two for a set of

* If we were certain that at all times the temperature of a bar is uniformly distributed over its whole length, one thermometer, if faultless, would be as good as three or four, or any larger number, nor would it matter at what point in the bar's length the temperature is so recorded. But, if we suppose that the temperature is liable to a gradual or uniform change from the one end of the bar to the other, we should feel constrained to place the one thermometer in the centre of the bar's length. If, with the same supposition as to the law of distribution of temperature, we had *two* thermometers, it would obviously suffice that they were equidistant from the extremities of the bar, and it would not matter whether they were either close to the two ends or close to the centre of the bar: in either case the mean of their indications would be the mean temperature of the bar, and would enable us to ascertain its length on the further supposition that every element Δx of the bar's length was expanded in precise accordance with its temperature. But suppose, as is more reasonable, that the temperature does not increase or diminish uniformly from one end to the other, but is expressed by such a law as $t = a_0 + a_1 x + a_2 x^2$, where a_2 is probably a very small quantity, and x is the distance of any point from the centre of the bar. Then the mean temperature of the bar is—

$$\tau = \int_{-\frac{1}{2}}^{+\frac{1}{2}} (a_0 + a_1 x + a_2 x^2) = a_0 + \frac{a_2}{12} \quad \ldots \ldots \quad (\alpha)$$

and the mean of temperatures indicated by two thermometers placed at equal distances $\pm i$ on either side of the centre is—

$$\tfrac{1}{2}(t_{+i} + t_{-i}) = a_0 + a_2 i^2 \quad \ldots \ldots \quad (\beta)$$

In order, then, that the mean of the two thermometers may give the mean temperature of the bar, we must have $(\alpha) = (\beta)$ or

$$i^2 = \tfrac{1}{12}$$

$$i = \pm \frac{1}{2\sqrt{3}}$$

This gives us the proper distance from the centre at which to place the thermometers. It is remarkable,—and in the case of a bar supported on two points, rather unfortunate too,—that these positions coincide with the two thermometers. In the thermometers of the first set, the degree is about 0·40 inch long, and is divided into tenths. These thermometers are constructed in pairs, one of each pair extending from 45° to 65°, and the other from 65° to 85°, each having one or two degrees in excess at either end: thus, when laid in their places on the bar, one pair have their bulbs close together at the centre of the left yard, the scales lying outwards; the second pair is similarly placed at the centre of the right yard, and the third pair have their bulbs close on each side of the centre of the bar. The second set of thermometers, a pair, extend each from about 30° to 105°, and the degrees are subdivided into halves only.

The method of determining the co-efficient of expansion, of which an instance is recorded at page 78, and all such methods, are open to the objection that the hot bar under observation is not in a state of repose but of change. The method of heating a bar with steam until it assumes under 212° a constant or apparently unchanging length, escapes from this objection, but is open to another as great or greater, viz., that the bar is done violence to and may not precisely return to its former length. If a bar of iron be heated from 62° to 212° it is so extended $150 \times 6 = 900$ millionths of its length; and if the modulus of elasticity be, say 30,000,000 and the section two square inches, the force required to produce the above extension is $60 \times 900 = 54{,}000$ lbs. or 24 tons. As a standard of length can seldom be used at a temperature exceeding 90°, it seems unnecessary that it should be heated in expansion experiments above 100° at the outside.

In the experiments to be recorded here, the hot bar was kept steadily at a constant temperature during each comparison, and the stability of the microscopes was not counted on further than in ordinary comparisons. The bars were compared, as follows:—

I_B cold, with I_S cold.
I_B hot, with I_S cold.
I_B cold, with I_S hot.
I_B hot, with I_S hot.

An irregular number, not generally exceeding four comparisons, were made each day. The experiments are divided into two series; the first extending from 17th February 1865 until the 17th of March; the second from the 29th of April to the 10th of May. In the first series the temperatures of the cold bar were from 39° to 46°, and of the hot from 74° to 99°; in the second series the cold bars were from 54° to 60°, and the hot from 75° to 96°. Inasmuch as they did not extend in temperature either high enough or low enough, the set of six thermometers were not used; but in their place the set of two at one-fourth and three-fourths the length of the bar. These thermometers were subjected to very careful comparisons which will be recorded further on.

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points at which the bar should be *supported*. Unfortunate, because the holes cut into the bar to admit the bulbs weaken the bar, whilst the greatest strain on the bar is just over the rollers. If there be four thermometers which we intend to place in the bar at equal distances, we shall find that in order to give the mean temperature of the bar, the thermometers should be placed so as to coincide with the points at which a bar carried on four rollers should, according to theory, be supported (*see* page 28). This, however, only reminds us that the theory which does not consider these cuttings is imperfect.

If there be three thermometers, one at the centre of the bar and the other two at distances $\pm i$ from the centre, then supposing still $t = a_0 + a_1 x + a_2 x^2$, the mean of their three indications is—

$$\tfrac{1}{3}(t_{-i} + t_0 + t_{+i}) = a_0 + \tfrac{2}{3} a_2 i^2 \; \ldots\ldots \quad (\gamma)$$

and in order that this may agree with the mean temperature of the bar

$$\tfrac{2}{3} i^2 = \tfrac{1}{12}$$

$$i = \pm \frac{1}{2\sqrt{2}}$$

which in a 10 ft. bar would indicate a position about 18 inches from either end.

In each comparison the bars were "observed" in the following order; supposing, for instance, one cold and the other hot:—

1. Cold bar.
2. Hot bar.
3. Hot bar.
4. Cold bar.

the bars being interchanged as rapidly as possible. Two observers always worked simultaneously. The following table shows the order and succession of the different readings forming an observation: A and B are the two observers:—

Order.	Micr. H	Left Thermr.	Right Thermr.	Micr. K	Order.
1	. . .	A	B	. . .	1
2	A	. . .	. . .	B	2
3	A	. . .	. . .	B	3
4	. . .	A	B	. . .	4
5	. . .	B	A	. . .	5
6	B	. . .	. . .	A	6
7	B	. . .	. . .	A	7
8	. . .	B	A	. . .	8

The two readings on each horizontal line are made simultaneously, and there are altogether four readings of each micrometer and four of each thermometer. This constitutes the "observing" of a bar in these expansion experiments, and contains 16 readings. Thus, each *comparison* involves 64 readings. Each bar lies between two long copper tanks full of water; in the case of the cold bar, the water and bar together take the (cold) temperature of the room; in the case of the hot bar, a current of hot water at a fixed temperature runs continually into the tanks and as continually runs out, provision being made that the tanks neither begin to empty nor to overflow. The water is heated outside of the building and is conveyed in through the wall by pipes; passing, in the interior of the room, through flexible tubes in order to allow the bars to be moved about and interchanged.

We shall now refer to the drawing, Plate IX, which is an isometric projection of the interior of the room, and explain the different parts of the apparatus.

This drawing shows the boxes of the two bars, one under the microscopes, the other in the middle of the room. These two boxes are the same in all respects.

aaaa . . . . . is a strong plank of mahogany, built up, about four inches in depth.
bbb'b' . . . . . two closed copper tanks extending the whole length of the mahogany plank, and towards either side of it.
cccc . . . . . . pieces of iron screwed to the plank, and holding the copper tanks steadily in their proper position.
dddd . . . . . vertical tubes communicating with the water in the tanks, allowing a thermometer to be inserted in each.
ee . . . . a vertical piece of brass tube secured below to the plank, open at its upper extremity. From it branch off
ff' . . . . . . tubes connecting it with
ggg'g' . . . . bent tubes connecting the two tanks.
h . . . . a flexible tube by which the supply of water enters *ee*, and passing through *ff'* and by *gg'* enters the two tanks, each tank in two places.

i . . . . a bent tube connecting the lower parts of the two tanks, and from whence the water is discharged through

h′ . . . . a flexible tube carrying away the water from the tanks, one at each end of the tanks.

i′ . . . . a tap regulating the amount of the discharge.

j . . . . a tube joining the two tanks at their upper surfaces and the middle of their length, causing the water of the two tanks to communicate.

j′ . . . . a bent tube connecting with and opening into *j*.

j″ . . . . a vertical tube into which *j′* opens. This tube is open above, and below communicates into

h″ . . . . a flexible tube carrying away the overflow of tanks.

kk′ . . . . . . two strong vertical plates of iron firmly screwed to the plank. There are two of these on each side of the box, and opposite one another.

ll′ . . . . cross bars of iron, supported by and screwed to, the upper extremities of *kk′*. The holes in *ll′* through which the screws pass are slotted, or enlarged, so that *ll′* admits of a slight adjustment in position. From *ll′* are suspended the two rollers which carry the bar.

mm′ . . . . . . screws whereby the rollers are elevated or depressed, and so the bar is made horizontal or the focal adjustment varied.

n . . . . the extremity of the bar.

oo′ . . . . apertures for reading the thermometers; each covered with a plate of glass.

pp′ . . . . . . thick plates of brass strongly screwed to the plank.

qqq′q′ . . . . . long steel screws working in the plates *pp′*, and which support on their points the whole weight of the box, bar, and water.

rr′ . . . . small iron trucks, each having two rollers, which receive the points of the screws *qqq′q′*, and consequently bear all the weight of box, bar, and water.

sss′s′ . . . . . mahogany rails on which the trucks run, and by which the box is run under the microscopes or removed from that position.

ttt′t′ . . . . . . posts carrying the front extremities of these rails.

uuu′u′ . . . . two strong frameworks of carpentry supporting the rear extremities of the rails, and screwed to the floor of the room.

vvv′v′ . . . . inner frames which slide vertically, and with the least possible friction, inside *uuu′u′*.

$s_{,}s_{,,}$ . . . . . . continuation of the rails *ss′*.

wwww . . . . counterpoise weights. There are eight weights in all, four to each sliding frame.

AAA . . . . the stone piers.

BBB . . . . . the mahogany beam shown in front view, Plate III.

The tanks are supplied with hot water through the flexible tube *h*. The water rising up in the tube *ee* flows along *f* and *f′* and each of these streams is again divided into two by the bent tubes *ggg′g′*. If the taps *i′* at each end of the box be left closed the water will, in overflowing, run out by the tube *j′j″*; and thence away by the flexible tube *h″*. But, in order that no part of the water may be *still*, and therefore liable to cool, it is allowed to run away by the flexible tubes *h′h′* from the extremities of the box. If the taps *i′* were left open the tanks would empty, and they have therefore to be so regulated that there shall be always water escaping by the overflow tube *h′*. This tube discharges into a lead pipe which crosses the passage *p* (Plate II.), and in the upper surface of this lead pipe is an orifice about two inches long by which the water can be seen running along the pipe; this orifice is kept under occasional observation, as, if there be no water passing through the pipe, the tanks may be emptying.

Thus, a continuous stream of water coming into the room from without, passes through the tanks, and thence out of the room again. This current has generally been kept up on the days of observation from 6 a.m. until 5 p.m. The tubes being flexible, there is no hindrance to the moving of the bars from one part of the room to another, running them (on their trucks *rr′*) along the rails *ss′*. The hot bar alone has a current of water passing through its tanks. Cold water might have been caused to pass continually through the tanks of the cold bar; but this was found unnecessary, as the cold bar is not materially influenced by the presence of the hot bar. Each box is supplied with the necessary tubing, as they are heated alternately. The same bar has remained in the same box during all the experiments.

It is of much importance that the bars be rapidly interchanged, and that without much exertion to the two observers who have to manipulate them; for the weight of each box when the tanks are full is 365 lbs. The boxes, when not under observation, rest, as will be seen in the drawing, on the points of the four screws *qqq′q′*. By driving these screws the box is of course raised, and by drawing the screws the box is let down. Resting by means of these screws on the little cast-iron trucks *rr′*, the boxes are moved with a very slight pressure, along the rails. When the box is brought under the microscopes, and so also over the carriages (marked *ggg′g′* in Plate III,) the screws *qqq′q′* are simultaneously worked by the two observers, one at each end of the box; and so the box, being lowered, takes its bearing on the carriages *ggg′g′*, and after a few more turns of the screws *qqq′q′*, is entirely free from the trucks. The trucks might now be removed, but they are left untouched so as to be ready to receive the screws' points when it is intended again to move the box. In order to remove the box from position under the microscopes, it must be taken off the carriages *ggg′g′*; this is effected simply by driving the screws *qqq′q′* (simultaneously, each observer working with both hands): immediately after the points come into contact with the trucks, the carriages are relieved of the weight which is transferred to the trucks, and the box is then rolled away.

But it is necessary for the interchange of two boxes that one pass the other. This is effected by lowering one of the boxes, that which has no current passing through the tanks, by means of the sliding frames *vvv′v′*. It will be seen that the upper part of this frame carries a rail which is in one and the same straight line with *ss′* so that there is no interruption to the passage of a bar over the sliding frame, but if, when the box is on this part of the rails, that is over the sliding frames, it receives a slight pressure from above it will begin to descend while the counterpoise weights commence to ascend. A very little pressure then lowers the box nearly to the floor of the room; this is very easily and rapidly effected by the two observers, one at each end. When the box is down there is of course an interruption of the rails; this is bridged over by a short piece of mahogany of the same section as the rails. Thus when the one box is down the other is free to pass from front to back of the room, or vice versâ.

Suppose now the cold bar has just been observed, and it is required to substitute for it the hot bar which is standing on the rear portion $s_{\prime}s_{\prime\prime}$ of the rails. The former is released from the carriages and transferred to bearings on its trucks; it is then run back, and on arriving at the sliding frame is allowed to go down to the floor; the bridge is then put on over the place of each sliding frame, and the hot bar is moved to the front and brought under the microscopes. This operation is effected with great facility.

When both bars are hot, the current of water must cease passing through the box that is lowered while it is in that position.

When either box is on the carriages *ggg′g′* it is of course perfectly under control for all adjustments transverse and longitudinal; the vertical adjustments being made by the screws *mm′*.

The cover of each bar box, by which the hot air is confined from rising upward or the cold air of the room from striking downward on the bar, is formed of several strips of mahogany half an inch thick, and covered with flannel, whose breadth is just equal to the distance apart of the tanks. These pieces are pressed down between the tanks, and so by the softness of the flannel fit nearly air-tight. The upper surface of these pieces is flush with the upper surface of the tanks. Apertures covered with glass are provided for the proper reading of the thermometers. This cover is interrupted just over each of the terminal lines of the bar, and an opening of about an inch left—extending across from tank to tank. Into this opening a small rectangular block of wood is fitted, whose under surface is very nearly, but not quite, in contact with the bar. A vertical cylindrical hole is bored in this piece of wood which exposes a small circular part of the surface of the bar to view, in the centre of which is the gold pin on which is drawn the line. Thus only a very small bit of the surface of the bar is exposed to the air in the reading of the lines. In order to allow the light of the candle to fall on the dot, the cylindrical hole just mentioned is cut away in a slope towards the light.

The boxes are open at the extremities; but to prevent the cold air rushing in when the tanks are hot, the ends are lightly packed with prepared cotton. The tanks are about half an inch longer than the bars at either extremity.

It is scarcely necessary to go into detail as to the mode of support of the bars further than to say (1) each bar was supported on two rollers, at the distance given by the formula, page 28; (2) each roller is held (by its axle) between two parallel rectangular plates of steel 8 inches long by 2 inches in width, framed together at the distance of 1·75 inch apart; (3) these plates swing on a knife edge carried by the crossbar ll', so that the roller is free to move in an arc of a circle, the axis of the roller describing a cylindrical surface whose radius is seven inches; (4) the knife edge is capable of being raised or lowered by the screws mm', and so the rollers are raised or lowered (*see* end view, fig. 5, Pl. X.). By this means the bar is at the most perfect liberty to expand or contract.

The mode of supply of hot water will be understood from Plate II.; α is a cylindrical cistern of sheet-iron, 37 inches in vertical length and 22½ inches in diameter; into it enter three pipes, one from the centre below vertically upward, another from above vertically downward, another from the side horizontally and reaching to the middle of the cylinder. Each of these pipes is furnished at its extremity with a "rose" like that of a watering-pot, the holes being fine and regular. By the first mentioned pipe hot water enters the cylinder; this water having been carried underground from a cylindrical boiler in an adjoining workshop. The upper pipe supplies cold water; β, γ, are taps holding in control the supplies of hot and cold water. The water in the boiler is heated some degrees above the heat required to be maintained in the reservoir α. The rose has the effect of preventing the hot water entering in as it were in a mass, it divides it into a multitude of fine streams which necessarily ascend, and so are constrained to mix generally with the water in the reservoir before reaching the top. Similarly, the rose attached to the supply pipe for cold water divides that supply into a number of fine streams whose tendency is to go to the bottom of the reservoir, thus getting well mixed with the hot water. The purpose of the rose on the third pipe, which passing through the walls of the bar room supplies the copper tanks, is, that water may be gathered from different points of the reservoir, so as to get the average temperature at those points. In order to show the temperature of the water that is escaping through the pipe into the bar room the bulb of a long thermometer is inserted into the rose. Any variation in the heat of the water is immediately shown by this thermometer. An assistant is told off to the duty of keeping this thermometer at a constant reading by regulating the taps $\beta\,\gamma$.

It is not possible to maintain a supply of water at a precisely uniform temperature; the thermometer will show oscillations, but these are easily kept at the *worst* within $\pm\,2^\circ\cdot5$ of the required temperature. These oscillations are performed in very short periods of time, as 30 seconds, so that it is very easy to make certain that the mean temperature of the water discharged in any period of five minutes shall be within a very small fraction of a degree of the required temperature. And it is to be remembered, that since the tanks take about 10 minutes to discharge, the small variations of temperature of the water which enters them, taking place as they do in very short intervals of time, do not produce similar variations of temperature in the tanks. Here the variations of temperature are very much slower and very much smaller, and in their influence on the bar generally insensible. Even with the existence of small sensible oscillations about a mean temperature there is this advantage above the method of observing a bar steadily cooling, that sometimes we observe the bar in the state of expanding and sometimes in the state of contracting, the one as often as the other, and thus a constant error is avoided; while however, at the same time we might *à priori* expect to get greater discordances than would be brought out in observing a bar steadily cooling.

During part of the observations the thermometer wells were filled with oil, during another part with mercury, and in the last series the air was simply excluded by packing round lightly over the bulbs with a very small piece of prepared cotton.

2.

The four thermometers 4219, 4226, 4222, 4220, used in the observations for the determination of the expansion of the bronze and steel bars have been compared with the Indian Standard Thermometer, No. 4142 (made by Casella).

For the determination of the errors of this Standard, (1) the boiling and freezing points were examined; (2) the calibration errors were determined for every five degrees from 32° to 97°; (3) the thermometer was compared with the Ordnance Standard Thermometer, No. 3241.

The readings of the boiling and freezing points were determined on the morning of March 22. The method of boiling the thermometer* has been described already. When the mercury had assumed a steady position under the influence of the steam, the manometer indicating no pressure, the thermometer was read four times at intervals of about five minutes. The following are the readings:—

	Reading.
By Quartermaster Steel, R.E.	212°·20
„ Captain Clarke „	212°·19
„ Quartermaster Steel „	212°·19
„ Captain Clarke „	212°·20

The thermometer was then removed from the steam, and having been allowed a few minutes to cool, was placed with its bulb in finely pounded ice. When the mercury had become stationary, readings were taken at intervals of about five minutes, the ice being removed and freshly adjusted round the bulb during the intervals of the reading. The results are as follow:—

	Readings.
By Quartermaster Steel	31°·99
„ Captain Clarke	32°·00
„ Quartermaster Steel	32°·00
„ Captain Clarke	32°·00

It remains only to remark that whether in the steam or the ice, the tube of the thermometer was in an accurately horizontal position.

The reading of the barometer at the time of the boiling of the thermometer was 29·912 in. (being corrected for index error and reduced to 32° Fahr.) The height of the

* The horizontal position of the thermometers when being boiled was adopted at the suggestion of Lieut.-Colonel Walker, Royal Engineers, Superintendent of the Great Trigonometrical Survey of India.

cistern was 4·50 feet above the thermometer; allowing for this, we have a pressure of 29·917 in. The *error*, therefore, of the thermometer at 212° is*—

$$212^\circ{\cdot}20 - 212^\circ + 1{\cdot}680\,(29{\cdot}905 - 29{\cdot}917)$$

or + 0°·18. Hence the correction to the reading of the thermometer at any temperature t on account of the error in the mean length of one degree is—

$$-\,0^\circ{\cdot}0010\,(t - 32^\circ)$$

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* The line 212° on Fahrenheit thermometers, or 100° on the Centigrade thermometers, represents the temperature of steam under *Laplace's* standard atmospheric pressure, or a barometric reading of 0·76$^{\text{metre}}$ (after correction to 32° F.) in the latitude of 45°. This pressure in any other latitude λ, will be represented by a column of mercury whose height is

$$0{\cdot}76\,\frac{\text{Gravity at latitude } 45^\circ}{\text{Gravity at latitude } \lambda}$$

$$= 0{\cdot}76\,\frac{1 + n\sin^2 45^\circ}{1 + n\sin^2\lambda}$$

$$= 0{\cdot}76\left(1 + \frac{n}{2}\cos 2\lambda\right)$$

here $n = \frac{5}{2}\,\frac{1}{289} - \frac{1}{294} = {\cdot}005250$, and 0·76 metre = 29·9215 inches, supposing the metre to be 39·3704 inches. The standard pressure then in latitude λ is—

$$29{\cdot}9215^{\text{in}}\,(1 + {\cdot}002625\cos 2\lambda)$$

But this supposes the barometer to be at the level of the sea. If the observation be made at some altitude above the sea, we must make allowance for the alteration of gravity arising from this cause. Suppose the observer stationed at the top of a hill, whose height is h (feet), and its form a cone with axis vertical. Let G be the force of gravity, or attraction of the earth, at the base of the cone, then if a be the radius of the earth, its attraction at the distance h above its surface

$$= G\left(1 - \frac{2h}{a}\right)$$

and to this we have to add the attraction of the conical hill. Now, the attraction A, of a cone whose vertical angle is 2α, the length of its axis h, and density δ, upon a particle at its apex, is

$$A = 2\pi h\delta\,(1 - \cos\alpha)$$

and supposing the hill to be of half the mean density of the earth we should have $G = \frac{8}{3}\pi a\delta$

$$\therefore\ \frac{A}{G} = \frac{3}{4}\,\frac{h}{a}\,(1 - \cos\alpha)$$

The total attraction therefore exercised upon the barometric column under the circumstances is

$$G\left(1 + \frac{3}{4}\,\frac{h}{a}\,(1 - \cos\alpha) - 2\,\frac{h}{a}\right)$$

If the place of observation be on an elevated *plane* at the height h above the sea, then $\alpha = 90^\circ$, and the attraction

$$= G\left(1 - \frac{5}{4}\,\frac{h}{a}\right)$$

The column of mercury, then, to indicate the same atmospheric pressure must be increased in the proportion of $1 : 1 + \frac{5}{4}\,\frac{h}{a}$; that is, the height of the column must be

$$= 29{\cdot}9215^{\text{in}}\left(1 + \frac{5}{4}\,\frac{h}{a} + {\cdot}002625\cos 2\lambda\right)$$

$$= 29{\cdot}9215^{\text{in}} + {\cdot}0785\cos 2\lambda + {\cdot}00000179\,h\dagger$$

According to the determination of Laplace from Dalton's experiments (*Mécanique Céleste*, Book X., Ch. 1.) the elastic force of vapour at 100° + i centigrade is—

$$F = 0{\cdot}76^{\text{m}}\,(10)^{0{\cdot}0154547\,i - 0{\cdot}0000625826\,i^2}$$

hence if μ be the modulus of the common system of logarithms,

$$\frac{\mu\,.\,dF}{F\,.\,di} = 0{\cdot}0154547$$

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† This at the equator and at the level of the sea = 30·0000 inches.

We next come to the process of calibration. This has already been described generally. A column of 60°·2 was broken off, and by it were compared the capacities of the tube from 32° to 92°, from 92° to 152°, and from 152° to 212°: these capacities we represent by [32·92], [92·152], [152·212]. The three spaces were compared five times. As the column is liable to variation of length from variation of temperature, which cannot be avoided, it is assumed that the column is not the same length in two different comparisons. In order that it may be safely assumed to remain constant during *one* comparison of the space, the observations are made and recorded as quickly as is consistent with accuracy. The following is the order of the readings: suppose the column so placed that its ends read *approximately* 31°·6 and 91°·8; then, (1) the line 31°, (2) the end of the column 31°·6, (3) the line 32°, are successively brought under the fixed wire of the microscope and the micrometer screw read, (4) the microscope being slid along its rails until the cross wire nearly bisects the line 91°,— this line, (5) the end of the column 91°·8, (6) the line 92° are successively brought under the cross wire of the microscope and the micrometer screw read. The thermometer remaining untouched, all these readings are repeated in the inverse order, viz., 92°, 91°·8, 91°; 32°, 31°·6, 31°.

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when $i = 0$. So that if δF be the increment of pressure corresponding to the increment of temperature δi, at the boiling point,

$$\delta F = \frac{0\cdot76}{\mu} 0\cdot0154547\, \delta i$$

$$\text{or } \delta F = \cdot02704\, \delta i$$

From the more recent investigations on this subject by M. V. Regnault (*Mémoires de l'Académie Royale des Sciences de l'Institut de France*, tom. xxi.), it appears that the elastic force of vapour is, from 0° to 100° centigrade, very accurately represented by the formula

$$\log. F = a + b\,\alpha^t - c\,\beta^t$$

where t is the temperature centigrade, and

$$\begin{aligned} \log. \alpha &= 0\cdot006865036 \\ \log. \beta &= \bar{1}\cdot9967249 \\ \log. b &= \bar{2}\cdot1340339 \\ \log. c &= 0\cdot6116485 \\ a &= 4\cdot7384380 \end{aligned}$$

F being expressed in millimetres. Here

$$\frac{dF}{dt} = \frac{F}{\mu^2} (b\,\alpha^t \log \alpha - c\,\beta^t \log. \beta)$$

the value of which when $t = 100$ is 27·217 millimetres, or 0·02722 of a metre which does not differ materially from the result obtained above from Laplace's formula.

From this, then, it appears that the variation of pressure corresponding to a variation of temperature, when the pressure is expressed in inches, and the temperature in degrees Fahrenheit, is

$$\delta F = \frac{0\cdot2722}{18} 39\cdot37\, \delta t = 0\cdot595\, \delta t$$

$$\text{or, } \delta t = 1\cdot680\, \delta F$$

Hence, if $\mathfrak{B}$ be the standard barometric pressure at the station of observation, B the actual height (reduced to 32°) of the barometer (differing but slightly from the standard height) at the time the thermometer is boiled then the temperature of the steam is

$$212° - 1\cdot680\,(\mathfrak{B} - B)$$

and if T be the reading of the thermometer

$$T - 212° + 1\cdot680\,(\mathfrak{B} - B)$$

is the error of the thermometer at the boiling point. At Southampton $\mathfrak{B} = 29\cdot9054$

The form in which the observations are recorded, is as follows:—

Part bisected.	Left extremity of Column.						Right extremity of Column.						Sum or Difference.
	Tube.	Micr. (1)	Micr. (4)	Mean.	Fraction.	Decimal.	Tube.	Micr. (2)	Micr. (3)	Mean.	Fraction.	Decimal.	
Line -	31°	209	208	209			91°	217	216	217			
Mercury -	31·6	440	439	440	$\frac{185}{416}$	·445	91·8	534	535	535	$\frac{102}{420}$	·243	·202
Line -	32	625	624	625			92	637	636	637			
Line -	91	211	211	211			151	208	208	208			
Mercury -	91·5	431	434	433	$\frac{197}{419}$	·470	151·7	516	516	516	$\frac{110}{418}$	·263	·207
Line -	92	630	630	630			152	627	625	626			
Line -	151	211	211	211			211	208	207	208			
Mercury -	151·3	338	335	337	$\frac{291}{417}$	·698	211·5	411	416	414	$\frac{218}{424}$	·514	·184
Line -	152	629	628	628			212	633	632	632			

The five comparisons of these three spaces give the following results:—

Space.	Equivalent.					
	Comp^n 1.	Comp^n 2.	Comp^n 3.	Comp^n 4.	Comp^n 5.	Mean.
[32·92]	C_1 −·195	C_2 −·206	C_3 −·171	C_4 −·217	C_5 −·202	C_0 −·198
[92·152]	C_1 −·204	C_2 −·214	C_3 −·185	C_4 −·217	C_5 −·207	C_0 −·205
[152·212]	C_1 −·165	C_2 −·196	C_3 −·151	C_4 −·183	C_5 −·184	C_0 −·176

In this table, C_1 C_2 C_3 C_4 C_5 are the lengths of the column in the corresponding comparisons; C_0 is the *mean* of these lengths. From this it appears, that—

$$[32 \cdot 92] = \tfrac{1}{3}\,[32 \cdot 212] - 0^\circ \cdot 005$$
$$[32 \cdot 152] = \tfrac{2}{3}\,[32 \cdot 212] - 0^\circ \cdot 017$$

The spaces [32·72], [42·82], [52·92] were next compared by means of a column of 40°; and the spaces [32·62], [42·72], [52·82], [62·92] by means of a column of 30°, the results are shown in the accompanying table:—

Space.	Equivalent.				
	Comp^n 1.	Comp^n 2.	Comp^n 3.	Comp^n 4.	Mean.
[32·72]	C_1 +·354	C_2 +·343	C_3 +·333	C_4 +·325	C_0 +·339
[42·82]	C_1 +·227	C_2 +·209	C_3 +·197	C_4 +·186	C_0 +·205
[52·92]	C_1 +·283	C_2 +·278	C_3 +·269	C_4 +·258	C_0 +·272
[32·62]	C'_1+·198	C'_2+·186	C'_3+·204	C'_4+·194	C'_0+·195
[42·72]	C'_1+·146	C'_2+·138	C'_3+·143	C'_4+·129	C'_0+·139
[52·82]	C'_1+·133	C'_2+·129	C'_3+·129	C'_4+·108	C'_0+·125
[62·92]	C'_1+·133	C'_2+·121	C'_3+·116	C'_4+·117	C'_0+·122

Now in order to determine the errors of the division lines 42°, 52°, 62°, 72°, 82°, let—

$$[32\cdot 42] = \tfrac{1}{6}\, m + x_4$$
$$[32\cdot 52] = \tfrac{2}{6}\, m + x_5$$
$$[32\cdot 62] = \tfrac{3}{6}\, m + x_6$$
$$[32\cdot 72] = \tfrac{4}{6}\, m + x_7$$
$$[32\cdot 82] = \tfrac{5}{6}\, m + x_8$$
$$[32\cdot 92] = m$$

then we get from the preceding comparisons, putting $C_0 + \cdot 200 = C$, $C'_0 + \cdot 100 = C'$,

$$\tfrac{4}{6}\, m + x_7 = C + \cdot 139$$
$$\tfrac{4}{6}\, m + x_8 - x_4 = C + \cdot 005$$
$$\tfrac{4}{6}\, m - x_5 = C + \cdot 072$$
$$\tfrac{3}{6}\, m + x_6 = C' + \cdot 096$$
$$\tfrac{3}{6}\, m + x_7 - x_4 = C' + \cdot 039$$
$$\tfrac{3}{6}\, m + x_8 - x_5 = C' + \cdot 025$$
$$\tfrac{3}{6}\, m - x_6 = C' + \cdot 022$$

Resolving these equations, we find—

$$x_4 = + \cdot 069$$
$$x_5 = + \cdot 018$$
$$x_6 = + \cdot 037$$
$$x_7 = + \cdot 049$$
$$x_8 = - \cdot 016$$

Again, by means of a column of 25°, the spaces [32·57], [37·62], [47·72], [57·82], [72·97], and [42·67], [52·77], [62·87], [67·92], [72·97] were compared. The results are shown in the following table:—

Space.	Equivalent.			
	Compⁿ 1.	Compⁿ 2.	Compⁿ 3.	Compⁿ 4.
[32·57]	$C_1 - \cdot 139$	$C_2 - \cdot 139$	$C_3 - \cdot 133$	$C_4 - \cdot 133$
[37·62]	$C_1 - \cdot 153$	$C_2 - \cdot 160$	$C_3 - \cdot 162$	$C_4 - \cdot 167$
[47·72]	$C_1 - \cdot 185$	$C_2 - \cdot 197$	$C_3 - \cdot 192$	$C_4 - \cdot 200$
[57·82]	$C_1 - \cdot 230$	$C_2 - \cdot 229$	$C_3 - \cdot 225$	$C_4 - \cdot 229$
[72·97]			$C_3 - \cdot 280$	$C_4 - \cdot 289$
[42·67]	$C'_1 - \cdot 203$	$C'_2 - \cdot 204$	$C'_3 - \cdot 191$	$C'_4 - \cdot 213$
[52·77]	$C'_1 - \cdot 218$	$C'_2 - \cdot 230$	$C'_3 - \cdot 214$	$C'_4 - \cdot 233$
[62·87]	$C'_1 - \cdot 205$	$C'_2 - \cdot 215$	$C'_3 - \cdot 205$	$C'_4 - \cdot 213$
[67·92]	$C'_1 - \cdot 178$	$C'_2 - \cdot 200$	$C'_3 - \cdot 199$	$C'_4 - \cdot 207$
[72·97]			$C'_3 - \cdot 279$	$C'_4 - \cdot 294$

Now in the first of the two series contained in this table, C is to be found from $\frac{1}{2}[32\cdot57]+\frac{1}{2}[57\cdot82]=\frac{1}{2}[32\cdot82]$ and so eliminated in each comparison. In the second series C′ is eliminated by getting its value from $\frac{1}{2}[42\cdot67]+\frac{1}{2}[67\cdot92]=\frac{1}{2}[42\cdot92]$. Thus we obtain, taking the mean of the comparisons—

$$[32\cdot57]=\tfrac{1}{2}[32\cdot82]+\cdot046$$
$$[37\cdot62]=\tfrac{1}{2}[32\cdot82]+\cdot022$$
$$[47\cdot72]=\tfrac{1}{2}[32\cdot82]-\cdot012$$
$$[72\cdot97]=\tfrac{1}{2}[32\cdot82]-\cdot105$$
$$[42\cdot67]=\tfrac{1}{2}[42\cdot92]-\cdot003$$
$$[52\cdot77]=\tfrac{1}{2}[42\cdot92]-\cdot024$$
$$[62\cdot87]=\tfrac{1}{2}[42\cdot92]-\cdot010$$
$$[72\cdot97]=\tfrac{1}{2}[42\cdot92]-\cdot084$$

But we have already seen that—

$$[32\cdot82]=\tfrac{5}{6}[32\cdot92]-\cdot016$$
$$[42\cdot92]=\tfrac{5}{6}[32\cdot92]-\cdot069$$

and these values are to be substituted in the preceding equations: making the substitution we get two different values for $[72\cdot97]$, viz. $=\frac{5}{12}[32\cdot92]-\cdot113$ and $=\frac{5}{12}[32\cdot92]-\cdot118$ of which we shall take the mean. Further we have found—

$$[32\cdot92]=\tfrac{1}{3}[32\cdot212]-\cdot005$$

And thus we are able to express the thirteen different spaces in terms of $[32\cdot212]$. The results are as follow:—

Standard Thermometer, 4142

$$[32\cdot37]=\tfrac{1}{12}[32\cdot92]+\cdot023=\tfrac{1}{36}[32\cdot212]+\cdot023$$
$$[32\cdot42]=\tfrac{2}{12}[32\cdot92]+\cdot069=\tfrac{2}{36}[32\cdot212]+\cdot068$$
$$[32\cdot47]=\tfrac{3}{12}[32\cdot92]+\cdot069=\tfrac{3}{36}[32\cdot212]+\cdot068$$
$$[32\cdot52]=\tfrac{4}{12}[32\cdot92]+\cdot018=\tfrac{4}{36}[32\cdot212]+\cdot016$$
$$[32\cdot57]=\tfrac{5}{12}[32\cdot92]+\cdot038=\tfrac{5}{36}[32\cdot212]+\cdot036$$
$$[32\cdot62]=\tfrac{6}{12}[32\cdot92]+\cdot037=\tfrac{6}{36}[32\cdot212]+\cdot035$$
$$[32\cdot67]=\tfrac{7}{12}[32\cdot92]+\cdot032=\tfrac{7}{36}[32\cdot212]+\cdot029$$
$$[32\cdot72]=\tfrac{8}{12}[32\cdot92]+\cdot049=\tfrac{8}{36}[32\cdot212]+\cdot046$$
$$[32\cdot77]=\tfrac{9}{12}[32\cdot92]-\cdot040=\tfrac{9}{36}[32\cdot212]-\cdot044$$
$$[32\cdot82]=\tfrac{10}{12}[32\cdot92]-\cdot016=\tfrac{10}{36}[32\cdot212]-\cdot020$$
$$[32\cdot87]=\tfrac{11}{12}[32\cdot92]-\cdot007=\tfrac{11}{36}[32\cdot212]-\cdot012$$
$$[32\cdot92]=\tfrac{12}{12}[32\cdot92]+\cdot000=\tfrac{12}{36}[32\cdot212]-\cdot005$$
$$[32\cdot97]=\tfrac{13}{12}[32\cdot92]-\cdot068=\tfrac{13}{36}[32\cdot212]-\cdot073$$

In the Ordnance Survey Standard Thermometer, No. 3241, the corresponding values obtained in precisely the same manner are—

Standard Thermometer, 3241.

$$[32\cdot37] = \tfrac{1}{12}[32\cdot92] + \cdot008 = \tfrac{1}{36}[32\cdot212] + \cdot003$$
$$[32\cdot42] = \tfrac{2}{12}[32\cdot92] - \cdot021 = \tfrac{2}{36}[32\cdot212] - \cdot031$$
$$[32\cdot47] = \tfrac{3}{12}[32\cdot92] - \cdot045 = \tfrac{3}{36}[32\cdot212] - \cdot060$$
$$[32\cdot52] = \tfrac{4}{12}[32\cdot92] - \cdot020 = \tfrac{4}{36}[32\cdot212] - \cdot041$$
$$[32\cdot57] = \tfrac{5}{12}[32\cdot92] - \cdot033 = \tfrac{5}{36}[32\cdot212] - \cdot059$$
$$[32\cdot62] = \tfrac{6}{12}[32\cdot92] - \cdot037 = \tfrac{6}{36}[32\cdot212] - \cdot068$$
$$[32\cdot67] = \tfrac{7}{12}[32\cdot92] - \cdot047 = \tfrac{7}{36}[32\cdot212] - \cdot083$$
$$[32\cdot72] = \tfrac{8}{12}[32\cdot92] - \cdot065 = \tfrac{8}{36}[32\cdot212] - \cdot106$$
$$[32\cdot77] = \tfrac{9}{12}[32\cdot92] - \cdot070 = \tfrac{9}{36}[32\cdot212] - \cdot116$$
$$[32\cdot82] = \tfrac{10}{12}[32\cdot92] - \cdot043 = \tfrac{10}{36}[32\cdot212] - \cdot095$$
$$[32\cdot87] = \tfrac{11}{12}[32\cdot92] + \cdot018 = \tfrac{11}{36}[32\cdot212] - \cdot039$$
$$[32\cdot92] = \tfrac{12}{12}[32\cdot92] + \cdot000 = \tfrac{12}{36}[32\cdot212] - \cdot062$$
$$[32\cdot97] = \tfrac{13}{12}[32\cdot92] + \cdot035 = \tfrac{13}{36}[32\cdot212] - \cdot032$$

For these data, a *curve of errors* is formed for each thermometer, the abscissa being the temperature t and the ordinate the error of the thermometer at the temperature t. Thirteen points are given, and a curve being drawn through them the errors of the thermometer at intermediate points is thus interpolated graphically.

In the Standard 3241 the correction for the error in the relative positions of the boiling and freezing point, or for the mean length of a degree, is $-0\cdot0010\ (t-32)$.

These two Standard Thermometers were compared together on the 17th April, commencing at the temperature 52° and ending with 97°. Immediately after the first comparisons the fire was lighted in a stove in the room, and the temperature of the room was made to increase regularly and continually so as to be nearly as possible the same as that of the water in the trough. It was not, however, found practicable to raise the temperature of the room to more than 90°. The thermometers were compared at or about 52°, 55°, 57°, 60°, 62°, 65° . . . 95°, 97°. In each position five comparisons were made, the thermometers lying close to one another and their tubes carefully made horizontal.

The following table contains the result of these observations, each line being the mean of five comparisons, except the last which is the mean of four.

No.	3241	4142	No.	3241	4142
	°	°		°	°
1	97·84	97·46	11	72·56	72·03
2	95·18	94·78	12	70·56	70·05
3	92·58	92·11	13	67·70	67·17
4	90·91	90·41	14	65·57	65·09
5	87·61	87·15	15	62·64	62·15
6	85·74	85·27	16	60·65	60·15
7	82·72	82·25	17	57·95	57·48
8	80·71	80·23	18	56·00	55·53
9	77·92	77·43	19	52·63	52·20
10	75·57	75·06			

Immediately after these comparisons the thermometers were placed in ice, and four different determinations of the freezing point made—with the following result:—

No.	3241	4142
	°	°
1	32·42	32·00
2	32·41	32·00
3	32·41	32·00
4	32·41	32·00

The total corrections then to the thermometer readings in the above table will be as subjoined—

No.	3241	4142	No.	3241	4142
	°	°		°	°
1	−·50	−·13	11	−·56	−·00
2	−·52	−·09	12	−·55	−·01
3	−·53	−·06	13	−·52	−·01
4	−·52	−·06	14	−·52	−·01
5	−·50	−·06	15	−·51	−·01
6	−·51	−·07	16	−·50	+·01
7	−·55	−·07	17	−·49	+·01
8	−·57	−·08	18	−·48	+·00
9	−·57	−·08	19	−·47	+·00
10	−·56	−·06			

Applying these corrections to the actual thermometer readings we get the following—

No.	3241	4142	No.	3241	4142
	°	°		°	°
1	97·34	97·33	11	72·00	72·03
2	94·66	94·69	12	70·01	70·04
3	92·05	92·05	13	67·18	67·16
4	90·39	90·35	14	65·05	65·08
5	87·11	87·09	15	62·13	62·14
6	85·23	85·20	16	60·15	60·16
7	82·17	82·18	17	57·46	57·49
8	80·14	80·15	18	55·52	55·53
9	77·35	77·35	19	52·16	52·20
10	75·01	75·00			

Taking now the true temperature to be the mean of those indicated by the two thermometers we have the following residual errors:—

Temperature.	3241	4142	Temperature.	3241	4142
°	°	°	°	°	°
97	+ 0·005	− 0·005	72	− 0·015	+ 0·015
95	− 0·015	+ 0·015	70	− 0·015	+ 0·015
92	0·000	0·000	67	+ 0·010	− 0·010
90	+ 0·020	− 0·020	65	− 0·015	+ 0·015
87	+ 0·010	− 0·010	62	− 0·005	+ 0·005
85	+ 0·015	− 0·015	60	− 0·005	+ 0·005
82	− 0·005	+ 0·005	57	− 0·015	+ 0·015
80	− 0·005	+ 0·005	56	− 0·005	+ 0·005
77	0·000	0·000	52	− 0·020	+ 0·020
75	+ 0·005	− 0·005			

The working thermometers 4219, 4226, 4222, 4220, were compared with 4142 on the 24th, 25th, and 27th of March. The following table contains the results; the number of comparisons of which each line is the mean, being shown in the last column:—

No. of Set.	4219	4226	4142	4222	4220	No. of Comparisons.
	°	°	°	°	°	
1	98·02	98·00	97·98	97·92	98·07	6
2	94·88	94·90	94·72	94·86	94·91	6
3	93·08	93·09	92·90	93·03	93·08	11
4	91·66	91·67	91·44	91·59	91·68	11
5	90·15	90·16	89·89	90·02	90·16	10
6	88·57	88·60	88·34	88·45	88·61	8
7	87·70	87·76	87·47	87·57	87·75	4
8	85·48	85·48	85·25	85·36	85·47	5
9	83·43	83·44	83·20	83·28	83·37	2
10	80·95	80·96	80·80	80·79	80·94	3
11	78·89	78·84	78·76	78·76	78·84	3
12	76·81	76·79	76·71	76·75	76·80	3
13	74·67	74·63	74·47	74·60	74·70	3
14	72·44	72·41	72·22	72·43	72·46	3
15	70·65	70·60	70·48	70·65	70·66	3
16	68·82	68·77	68·64	68·83	68·87	3
17	67·25	67·21	67·11	67·26	67·27	3
18	65·40	65·35	65·25	65·41	65·43	3
19	64·50	64·43	64·35	64·50	64·52	3
20	62·19	62·19	62·09	62·18	62·24	3
21	61·25	61·20	61·11	61·22	61·29	3
22	59·80	59·71	59·66	59·76	59·84	3
23	57·65	57·53	57·50	57·59	57·68	3
24	55·82	55·74	55·68	55·75	55·82	3
25	54·58	54·50	54·42	54·50	54·58	2
26	51·97	51·88	51·80	51·89	51·91	3
27	48·42	48·30	48·17	48·31	48·30	3
28	46·55	46·46	46·27	46·45	46·51	3
29	44·42	44·34	44·15	44·25	44·41	3
30	43·39	43·32	43·13	43·27	43·40	3
31	42·44	42·38	42·21	42·33	42·46	3
32	41·72	41·69	41·52	41·63	41·79	3

On the conclusion of these comparisons the thermometers were all placed in ice, and the means of six determinations of the freezing point of each thermometer were as follow:—

4219	4226	4142	4222	4220
32°·11	32°·10	32°·00	31°·98	32°·11

If now we correct the readings in the preceding table for the errors of the freezing points, and, in the case of the Standard 4142, apply also the corrections for calibration and boiling point errors, we get the numbers shown below:—

No. of Set.	4219	4226	4142	4222	4220
1	97°·91	97°·90	97°·84	97°·94	97°·96
2	94·77	94·80	94·63	94·88	94·80
3	92·97	92·99	92·83	93·05	92·97
4	91·55	91·57	91·38	91·61	91·57
5	90·04	90·06	89·83	90·04	90·05
6	88·46	88·50	88·27	88·47	88·50
7	87·59	87·66	87·40	87·59	87·64
8	85·37	85·38	85·17	85·38	85·36
9	83·32	83·34	83·12	83·30	83·26
10	80·84	80·86	80·73	80·81	80·83
11	78·78	78·74	78·67	78·78	78·73
12	76·70	76·69	76·62	76·77	76·69
13	74·56	74·53	74·44	74·62	74·59
14	72·33	72·31	72·23	72·45	72·35
15	70·54	70·50	70·49	70·67	70·55
16	68·71	68·67	68·63	68·85	68·76
17	67·14	67·11	67·10	67·28	67·16
18	65·29	65·25	65·24	65·43	65·32
19	64·39	64·33	64·34	64·52	64·41
20	62·08	62·09	62·09	62·20	62·13
21	61·14	61·10	61·12	61·24	61·18
22	59·69	59·61	59·67	59·78	59·73
23	57·54	57·43	57·51	57·61	57·57
24	55·71	55·64	55·68	55·77	55·71
25	54·47	54·40	54·42	54·52	54·47
26	51·86	51·78	51·80	51·91	51·80
27	48·31	48·20	48·21	48·33	48·19
28	46·44	46·36	46·32	46·47	46·40
29	44·31	44·24	44·22	44·27	44·30
30	43·28	43·22	43·19	43·29	43·29
31	42·33	42·28	42·27	42·35	42·35
32	41·61	41·59	41·57	41·65	41·68

Referring now the other thermometers to the Standard 4142, we get the following final system of errors, which must be regarded as calibration errors :—

Temp.	Bronze Bar. Errors of 4219	Bronze Bar. Errors of 4226	Bronze Bar. Error of Mean.	Steel Bar. Errors of 4222	Steel Bar. Errors of 4220	Steel Bar. Error of Mean.	Temp.
98°	+0·07	+0·06	+0·07	+0·10	+0·12	+0·11	98°
95	+0·14	+0·17	+0·16	+0·25	+0·17	+0·21	95
93	+0·14	+0·16	+0·15	+0·22	+0·14	+0·18	93
92	+0·17	+0·19	+0·18	+0·23	+0·19	+0·21	92
90	+0·21	+0·23	+0·22	+0·21	+0·22	+0·22	90
89	+0·19	+0·23	+0·21	+0·20	+0·23	+0·22	89
88	+0·19	+0·26	+0·23	+0·19	+0·24	+0·22	88
85	+0·20	+0·21	+0·21	+0·21	+0·19	+0·20	85
83	+0·20	+0·22	+0·21	+0·18	+0·14	+0·16	83
81	+0·11	+0·13	+0·12	+0·08	+0·10	+0·09	81
79	+0·11	+0·07	+0·09	+0·11	+0·06	+0·09	79
77	+0·08	+0·07	+0·08	+0·15	+0·07	+0·11	77
75	+0·12	+0·09	+0·11	+0·18	+0·15	+0·17	75
72	+0·10	+0·08	+0·09	+0·22	+0·12	+0·17	72
71	+0·05	+0·01	+0·03	+0·18	+0·06	+0·12	71
69	+0·08	+0·04	+0·06	+0·22	+0·13	+0·18	69
67	+0·04	+0·01	+0·03	+0·18	+0·06	+0·12	67
65	+0·05	+0·01	+0·03	+0·19	+0·08	+0·14	65
64	+0·05	−0·01	+0·02	+0·18	+0·07	+0·13	64
62	−0·01	0·00	−0·01	+0·11	+0·04	+0·08	62
61	+0·02	−0·02	0·00	+0·12	+0·06	+0·09	61
60	+0·02	−0·06	−0·02	+0·11	+0·06	+0·09	60
58	+0·03	−0·08	−0·03	+0·10	+0·06	+0·08	58
56	+0·03	−0·04	−0·01	+0·09	+0·03	+0·06	56
55	+0·05	−0·02	+0·02	+0·10	+0·05	+0·08	55
52	+0·06	−0·02	+0·02	+0·11	0·00	+0·06	52
48	+0·10	−0·01	+0·05	+0·12	−0·02	+0·05	48
47	+0·12	+0·04	+0·08	+0·15	+0·08	+0·12	47
44	+0·09	+0·02	+0·06	+0·05	+0·08	+0·07	44
43	+0·09	+0·03	+0·06	+0·10	+0·10	+0·10	43
42	+0·06	+0·01	+0·04	+0·08	+0·08	+0·08	42
41	+0·04	+0·02	+0·03	+0·08	+0·11	+0·10	41

To find the actual errors of these thermometers at any given time, it is necessary to *add* to the quantities in this Table the error of the freezing point in each thermometer, which is *positive* when the thermometer placed in melting ice reads *higher* than 32°·00.

On April 27, just before the commencement of the *second* series of expansion experiments, the freezing points of the thermometers were found as follow :—

4219	4226	4222	4220
32°·14	32°·10	32°·08	32°·12

3.

The actual expansion of a 10-feet bar of bronze for 60° Fahrenheit is about the sixteenth part of an inch, and this is a large quantity for micrometer measurement; it is, however, divided between two micrometers, so that either of them only measures half this amount. The actual number of divisions measured in several of the comparisons is upwards of 1000 divisions by each microscope, and in one case the quantity exceeds 1100 divisions; so that 11 revolutions of the screw are brought into play. In all our preceding operations no quantities of this magnitude have been measured, and it becomes necessary to have some assurance that the screws of the micrometers have no irregularities which would vitiate our results.

In order to put this to the test, three contiguous or successive spaces of one hundredth of an inch on the foot **OF** were selected; viz., the three right-hand spaces in the tenth [6·7]. Referring to page 58 it will be seen that those three spaces are not quite equal, but taking the mean of the results given by the two microscopes, if 3 S be the sum of the three spaces

$$\begin{aligned} \text{Left space} &= S + 0\overset{d}{\cdot}75 \\ \text{Centre space} &= S - 1\cdot58 \\ \text{Right space} &= S + 0\cdot83 \end{aligned} \qquad (1)$$

The three spaces were then measured in different parts of the field. The observations were all made by one observer, and for each microscope the focal adjustment remained constant during the operation. First, the scale was adjusted (being mounted as in fig 5, Plate VI.) so that (in the microscope H) the apparent right-hand line of the four which contain the three spaces, read approximately, 300·0. The scale remaining untouched, two readings of this line are made, the micrometer screw revolved until the next line is reached, which is bisected twice. The movement of the screw being continued, the third, and then the fourth lines, are each twice bisected. After the two bisections of the fourth, or apparent left-hand line, the operation is reversed; the fourth line is again bisected twice, then the third line the same, the second, and finally the first. As the scale **OF** remains untouched during this time, the second readings should differ from the first only by errors of observation. The readings of the four lines are 300, 650, 1000, and 1350, very closely.

The scale is then moved by the motion of the carriages *gg* until the right-hand line reads 350; and with this setting, the operation above recorded is repeated. The reading of the left-hand line is in this position 1400.

The same operation is repeated, commencing with 400, 450, 500, 550, 600, 650; the reading of the left-hand line in the last case being 1700. The *centre* of the field reads 1000.

Thus the three spaces are measured in eight different positions in the field, differing half a revolution each.

The whole of this is repeated five times over for each micrometer.

Taking first the microscope H, the mean results are as follow:

d	d	d	d	
1353·06	1002·34	652·10	300·54	(2)
1403·18	1051·50	701·54	349·70	
1454·24	1102·48	752·40	400·42	
1503·84	1151·96	802·22	450·10	
1553·24	1201·58	851·96	500·50	
1604·08	1252·42	902·78	550·96	
1652·44	1301·00	951·36	599·48	
1702·88	1351·38	1001·98	650·12	

In order that the differences here may represent the same quantity we must correct for the inequality of the three spaces; that is, the second column (vertical) must have the quantity $+ 0 \cdot 83$ applied, and the third column the quantity $- 0 \cdot 75$. Supposing this done, let the irregularities of the screw be such as to require the corrections

$$\begin{array}{lll} + x_{30} & \text{to the reading} & 300 \\ + x_{35} & \text{,,} \quad \text{,,} & 350 \\ \vdots & & \vdots \\ + x_{170} & \text{,,} \quad \text{,,} & 1700 \end{array} \tag{3}$$

And let $350 + h$ be the real value of the interval being measured, or the mean value of the three spaces.

Take now the differences of the first readings corrected, and we get

$$\begin{aligned} 1052 \cdot 52 + x_{135} - x_{30} &= 1050 + 3h \\ 702 \cdot 63 + x_{100} - x_{30} &= 700 + 2h \\ 350 \cdot 81 + x_{65} - x_{30} &= 350 + h \end{aligned}$$

or

$$\begin{aligned} x_{135} - x_{30} - 3h + 2 \cdot 52 &= 0 \\ x_{100} - x_{30} - 2h + 2 \cdot 63 &= 0 \\ x_{65} - x_{30} - h + 0 \cdot 81 &= 0 \end{aligned} \tag{4}$$

So each of the other lines will give three equations, making 24 in all, from which we have to get 29 x's and h. So that the problem is indeterminate in this shape.

Each of the numbers in (2) is the mean of 10 readings; it is therefore liable to the probable error of $\frac{0 \cdot 32}{\sqrt{10}} = \pm 0^{d} \cdot 10$ from the inevitable errors of observation. It is not then quite correct to throw all these errors upon the screw. Indeed, to determine with anything like precision the errors of the screw, a vastly greater number of observations would be required; but our object is rather to show that the errors, whatever they be are very small.

We shall proceed in the following manner: writing out all the discrepancies of the observations, as we have done for the first line, but without turning them into equations as (4), we shall determine that system of corrections $x_{30} \ldots\ldots x_{170}$ which shall make the sum of the squares of the discrepancies, together with the sum of the squares of the corrections themselves, a minimum. That is

$$\begin{aligned} & (-x_{30} + x_{65} - h + 0 \cdot 81)^2 \\ + & (-x_{30} + x_{100} - 2h + 2 \cdot 63)^2 \\ + & (-x_{30} + x_{135} - 3h + 2 \cdot 52)^2 \\ + & (-x_{65} + x_{100} - h + 1 \cdot 11)^2 \\ + & (-x_{65} + x_{135} - 2h + 2 \cdot 00)^2 \\ + & (-x_{65} + x_{170} - 3h + 2 \cdot 76)^2 \\ + & (-x_{35} + x_{70} - h + 1 \cdot 09)^2 \\ + & (-x_{35} + x_{105} - 2h + 2 \cdot 63)^2 \\ + & (-x_{35} + x_{140} - 3h + 3 \cdot 48)^2 \\ + & (-x_{40} + x_{75} - h + 1 \cdot 23)^2 \\ + & (-x_{40} + x_{110} - 2h + 2 \cdot 89)^2 \\ + & (-x_{40} + x_{145} - 3h + 3 . 83)^2 \\ + & (-x_{45} + x_{80} - h + 1 \cdot 37)^2 \end{aligned} \tag{5}$$

$$
\begin{aligned}
&+ (-x_{45} + x_{115} - 2h + 2\cdot 69)^2 \\
&+ (-x_{45} + x_{150} - 3h + 3\cdot 74)^2 \\
&+ (-x_{50} + x_{85} - h + 0\cdot 71)^2 \\
&+ (-x_{50} + x_{120} - 2h + 1\cdot 91)^2 \\
&+ (-x_{50} + x_{155} - 3h + 2\cdot 74)^2 \\
&+ (-x_{55} + x_{90} - h + 1\cdot 07)^2 \\
&+ (-x_{55} + x_{125} - 2h + 2\cdot 29)^2 \\
&+ (-x_{55} + x_{160} - 3h + 3\cdot 12)^2 \\
&+ (-x_{60} + x_{95} - h + 1\cdot 13)^2 \\
&+ (-x_{60} + x_{130} - 2h + 2\cdot 35)^2 \\
&+ (-x_{60} + x_{165} - 3h + 2\cdot 96)^2 \\
&+ (x^2_{30} + x^2_{35} + \ldots . x^2_{105} + x^2_{170})
\end{aligned}
$$

must be a minimum by the variation of $x_{30}\ x_{35}\ \ldots\ x_{170}$ and h. The solution is simple as the quantities x are not much intermixed; only $x_{65}\ x_{100}\ x_{135}$, occur twice in the discrepancies, we have therefore placed together the discrepancies obtained from the first and last lines of (2); they form the first six of the quantities just written down.

Differentiating first with respect to $x_{30}\ x_{65}\ x_{100}\ x_{135}\ x_{170}$ we get

$$
\begin{aligned}
4\,x_{30} - x_{65} - x_{100} - x_{135} \phantom{- x_{170}} - 5\cdot 96 + 6\,h &= 0 \\
-\,x_{30} + 5\,x_{65} - x_{100} - x_{135} - x_{170} - 5\cdot 15 + 5\,h &= 0 \\
-\,x_{30} - x_{65} + 3\,x_{100} \phantom{- x_{135}} + 3\cdot 74 - 3\,h &= 0 \\
-\,x_{30} - x_{65} + 3\,x_{135} + 4\cdot 61 - 5\,h &= 0 \\
-\,x_{65} + 2\,x_{170} + 2\cdot 76 - 3\,h &= 0
\end{aligned}
$$

whence the quantities x are easily expressed in terms of h.

Again, differentiating with respect to $x_{35}\ x_{70}\ x_{105}\ x_{140}$

$$
\begin{aligned}
4\,x_{35} - x_{70} - x_{105} - x_{140} - 7\cdot 20 + 6\,h &= 0 \\
-\,x_{35} + 2\,x_{70} + 1\cdot 09 - h &= 0 \\
-\,x_{35} + 2\,x_{105} + 2\cdot 63 - 2\,h &= 0 \\
-\,x_{35} + 2\,x_{140} + 3\cdot 48 - 3\,h &= 0
\end{aligned}
$$

Here again the quantities x are easily expressed in terms of h; and so on. Finally we have to differentiate with respect to h, which gives one equation containing all the x's.

When the necessary substitutions are made, the equation becomes

$$- 29\cdot 432 + 27\cdot 15\,h = 0$$
$$\therefore h = 1\cdot 084$$

and now all the corrections x are numerically known. Their values are as follow :— (6)

$x_{30} = -\cdot 15$	$x_{80} = -\cdot 01$	$x_{130} = -\cdot 10$
$x_{35} = +\cdot 14$	$x_{85} = +\cdot 07$	$x_{135} = +\cdot 20$
$x_{40} = +\cdot 29$	$x_{90} = +\cdot 00$	$x_{140} = -\cdot 05$
$x_{45} = +\cdot 26$	$x_{95} = -\cdot 03$	$x_{145} = -\cdot 14$
$x_{50} = -\cdot 23$	$x_{100} = -\cdot 23$	$x_{150} = -\cdot 12$
$x_{55} = -\cdot 00$	$x_{105} = -\cdot 16$	$x_{155} = +\cdot 14$
$x_{60} = -\cdot 01$	$x_{110} = -\cdot 22$	$x_{160} = +\cdot 06$
$x_{65} = -\cdot 04$	$x_{115} = -\cdot 13$	$x_{165} = +\cdot 14$
$x_{70} = +\cdot 07$	$x_{120} = +\cdot 02$	$x_{170} = +\cdot 22$
$x_{75} = +\cdot 07$	$x_{125} = -\cdot 06$	

Of these quantities 14 are greater than $0^d\cdot 12$, and 15 less than $0^d\cdot 12$.

When these corrections are applied to the micrometer readings they leave the following discrepancies, corresponding to the lines in (2)

− ·38	+ ·39	− ·16	(7)
+ ·04	+ ·16	− ·06	
+ ·14	+ ·21	− ·07	
+ ·11	+ ·13	+ ·02	
− ·15	− ·01	− ·07	
− ·07	+ ·06	− ·01	
− ·14	+ ·09	+ ·03	
− ·23	+ ·16	− ·16	

Of these, twelve are greater, and twelve less than $0^d \cdot 12$. Now the probable error of any one of the numbers in (2) being $\pm\, 0^d \cdot 10$, the probable error of the difference of two of these numbers is $\pm\, 0^d . 14$; so that the residual discrepancies after making use of the corrections x to the micrometer readings are somewhat less than might have been expected.

It appears, then, that in the microscope H there is nothing to be feared from irregularity of the screw within the limits up to which it has been used, or within seven revolutions on either side of the centre of the field.

The same method of observation gave for the microscope K the following numbers—

d	d	d	d	
300·48····	652·02····	1000·12····	1350·48	(8)
350·18····	701·70····	1049·40····	1399·74	
400·26····	751·70····	1099·58····	1450·00	
450·66····	801·84····	1149·78····	1499·68	
499·12····	849·68····	1197·78····	1547·82	
549·46····	900·58····	1248·32····	1598·24	
600·12····	951·34····	1299·10····	1649·40	
650·04····	1001·00····	1348·66····	1699·04	

In order to correct for inequality of the spaces, the numbers in the second vertical column must have $-\, 0^d \cdot 83$ applied, and the numbers in the third column $+\, 0^d \cdot 75$.

Proceeding now as in the case of microscope H, we get for h the equation

$$7 \cdot 344 + 27 \cdot 15\, h = 0$$
$$\therefore\; h = -\, 0 \cdot 270$$

and the values of the quantities x are found as follow:—

$x_{30} = +\, \cdot 56$	$x_{80} = -\, \cdot 22$	$x_{130} = -\, \cdot 03$	(9)
$x_{35} = +\, \cdot 37$	$x_{85} = -\, \cdot 05$	$x_{135} = -\, \cdot 11$	
$x_{40} = +\, \cdot 41$	$x_{90} = -\, \cdot 25$	$x_{140} = -\, \cdot 00$	
$x_{45} = +\, \cdot 17$	$x_{95} = -\, \cdot 23$	$x_{145} = -\, \cdot 07$	
$x_{50} = -\, \cdot 11$	$x_{100} = -\, \cdot 31$	$x_{150} = +\, \cdot 17$	
$x_{55} = +\, \cdot 06$	$x_{105} = -\, \cdot 07$	$x_{155} = +\, \cdot 19$	
$x_{60} = +\, \cdot 20$	$x_{110} = -\, \cdot 10$	$x_{160} = +\, \cdot 23$	
$x_{65} = -\, \cdot 16$	$x_{115} = -\, \cdot 12$	$x_{165} = +\, \cdot 06$	
$x_{70} = -\, \cdot 30$	$x_{120} = -\, \cdot 03$	$x_{170} = +\, \cdot 02$	
$x_{75} = -\, \cdot 24$	$x_{125} = -\, \cdot 04$		

Of these corrections, fourteen are less than $0^d \cdot 14$ and fifteen exceed that quantity.

If now we correct the micrometer readings by these quantities and then take out the differences, the following discrepancies remain. They are arranged to correspond with the lines in (8).

+ ·26	+ ·06	+ ·14	(10)
+ ·29	+ ·07	+ ·00	
+ ·23	+ ·10	+ ·07	
+ ·23	+ ·12	− ·17	
+ ·06	+ ·03	− ·19	
+ ·25	+ ·05	− ·24	
+ ·23	+ ·04	− ·05	
+ ·25	− ·04	− ·01	

Twelve of these errors exceed $0^d·11$ and twelve are less than that quantity. There is, however, a preponderance of + quantities in the first column. This may be due to the circumstance that the equations (1) are not free from probable error. In fact, there is a discordance between the results given by H and K, page 58, as to the relative magnitude of the three spaces.

We have now tolerably satisfactory evidence that there is nothing to fear from the irregularities of the micrometer screws.

The value of a revolution of either of the screws which has been used up to this time, was determined from measurement of a space of one hundredth of an inch or 350 divisions, and since no quantities measured hitherto have been so great as this, the determination has sufficed. It becomes necessary now, however, to obtain the value of a revolution from the measurement of a much larger space. Accordingly a space of four hundredths of an inch was chosen, being the four hundredths adjoining the line 6 on the foot **OF**. Now the value of this space is (see page 59)—

$$s = \tfrac{4}{10}\,[6·7] - 2^d·00 \pm 0^d·104$$

$$\text{also} \quad [6·7] = \tfrac{1}{10}\,\mathrm{I} - 1^d·78 \pm 0^d·092$$

$$\therefore \tfrac{4}{10}\,[6·7] = \tfrac{4}{100}\,\mathrm{I} - 0^d·71 \pm 0^d·037$$

$$\therefore s = \tfrac{4}{100}\,\mathrm{I} - 2^d·71 \pm 0^d·110 \qquad (11)$$

The space was measured ten times by each of three observers:—Captain Clarke, R.E., Quartermaster Steel, and Corporal Compton. The scale was freshly adjusted to focus each time so as *fully* to bring out all error arising from imperfect focusing. In each measure each line was bisected twice, and the mean of the two readings taken. The following table shows the individual measures—

Microscope H.			Microscope K.		
d	d	d	d	d	d
1393·2	1393·1	1393·5	1387·8	1387·8	1388·4
1393·9	1393·0	1393·0	1388·3	1388·8	1388·7
1393·3	1393·8	1394·4	1387·6	1387·6	1389·1
1393·2	1393·3	1393·6	1388·8	1387·9	1387·2
1393·8	1394·3	1393·7	1388·4	1389·7	1388·5
1393·0	1392·3	1392·8	1388·9	1389·6	1389·4
1393·3	1394·3	1394·2	1388·3	1387·3	1388·9
1394·0	1395·0	1394·4	1388·6	1387·3	1388·5
1395·4	1393·4	1393·8	1388·3	1391·4	1388·8
1394·0	1395·4	1394·1	1388·4	1388·2	1388·1

For microscope H the mean results by the different observers are 1393·71, 1393·79, and 1393·75, while the general mean is 1393·75. For microscope K the individual means are 1388·34, 1388·56, and 1388·56, and the general mean 1388·49. Taking the general mean in each case as the truth, we get the following errors :—

Microscope H.			Microscope K.		
d	d	d	d	d	d
− 0·55	− 0·65	− 0·25	− 0·69	− 0·69	− 0·09
+ 0·15	− 0·75	− 0·75	− 0·19	+ 0·31	+ 0·21
− 0·45	+ 0·05	+ 0·65	− 0·89	− 0·89	+ 0·61
− 0·55	− 0·45	− 0·15	+ 0·31	− 0·59	− 1·29
+ 0·05	+ 0·55	− 0·05	− 0·09	+ 1·21	+ 0·01
− 0·75	− 1·45	− 0·95	+ 0·41	+ 1·11	+ 0·91
− 0·45	+ 0·55	+ 0·45	− 0·19	− 1·19	+ 0·41
+ 0·25	+ 1·25	+ 0·65	+ 0·11	− 1·19	+ 0·01
+ 1·65	− 0·35	+ 0·05	− 0·19	+ 2·91	+ 0·31
+ 0·25	+ 1·65	+ 0·35	− 0·09	− 0·29	− 0·39

If $\pm\ 0^d\cdot 32$ be taken as the probable error of a single reading, then the probable error of a single measurement, as described above, so far as is due to bisections only will also be $\pm\ 0^d\cdot 32$. The errors of the first and third observers (shown in the first and third columns under each microscope) do not indicate any increase of magnitude due to focusing, but the errors of the second observer do. We shall assume that this disposition of the errors is owing to chance. The sum of the squares of the thirty errors for H is 15·475, whence the probable error of one measurement is—

$$\pm\ \cdot 674 \sqrt{\frac{15\cdot 475}{30 - 1}} = \pm\ \cdot 492 \qquad (12)$$

and the probable error of the general mean

$$\pm\ \frac{\cdot 492}{\sqrt{30}} = \pm\ 0^d\cdot 090$$

For K the sum of the squares of the errors is 20·795 whence the probable error of one measure is

$$\pm\ \cdot 674 \sqrt{\frac{20\cdot 795}{30 - 1}} = \pm\ \cdot 571 \qquad (13)$$

and the probable error of the general mean

$$\pm\ \frac{\cdot 571}{\sqrt{30}} = \pm\ 0^d\cdot 104$$

so that the number of divisions of the two micrometers corresponding to the space in question, are—

$$H \ldots\ldots 1393^d\cdot 75 \pm 0^d\cdot 090$$
$$K \ldots\ldots 1388^d\cdot 49 \pm 0^d\cdot 104$$

These quantities are equivalent to $\frac{4}{100}\ I - 2^d\cdot 71\ \pm\ 0^d\cdot 110$

$$\therefore\ (1396\cdot 46 \pm 0\cdot 14)\ h = \tfrac{4}{100}\ I.$$
$$(1391\cdot 20 \pm 0\cdot 15)\ k = \tfrac{4}{100}\ I.$$

and expressing I in millionths of a yard

$$h = \frac{1111 \cdot 11}{1396 \cdot 46 \pm \cdot 14} = 0 \cdot 79566 \pm \cdot 00008 \qquad (14)$$

$$k = \frac{1111 \cdot 11}{1391 \cdot 20 \pm \cdot 15} = 0 \cdot 79867 \pm \cdot 00009 \qquad (15)$$

These results differ from those obtained at page 64 by one division in 1100 and one division in 1200 respectively.

They differ also as to the amount of error of measure apparently due to focal error of adjustment. Here we have the probable error due to focal adjustment on a measure of 1390 divisions equal to

$$\text{for H} \ldots\ldots \pm \sqrt{(\cdot 492)^2 - (\cdot 32)^2} = \pm \cdot 374$$

$$\text{for K} \ldots\ldots \pm \sqrt{(\cdot 571)^2 - (\cdot 32)^2} = \pm \cdot 473$$

which gives for a measure of one thousand divisions a probable error

$$\varepsilon_H = \pm 0 \cdot 269$$

$$\varepsilon_K = \pm 0 \cdot 340$$

These quantities are smaller than those obtained at page 63.

4.

We shall now give the results of the different comparisons in the following table. The first two columns contain the day and hour: the third and fourth, the observed micrometer readings and temperature of the bronze bar: the fifth column the corrections for errors of thermometers. The sixth, seventh, and eighth give the micrometer and thermometer readings, and the corrections to the latter, for the steel bar. The lines are arranged in pairs: the first line of each pair contains the mean readings of one observer, Captain Clarke, R.E.; the second, the mean of simultaneous readings of the second observer, Quartermaster Steel, R.E. From the general description which has been given of the mode of observing, it will be understood that each number in the third and sixth columns is the mean of four micrometer readings, and each number in the fourth and seventh columns the mean of eight readings, viz., four readings of the left thermometer and four readings of the right thermometer.

Date.		Bronze Bar.			Steel Bar.			No. of Comparison.
Day.	Hour.	Micrometer Measurements.	Obs^d Temp.	Corr^n to Therm^s.	Micrometer Measurements.	Obs^d Temp.	Corr^n to Therm^s.	
1865.	H. M.		°			°		
Feb. 17	4 30p.m.	$1023 \cdot 23h + 1034 \cdot 73k$	38·82	−0·13	$945 \cdot 98h + 948 \cdot 83k$	38·75	−0·08	1
		$1026 \cdot 80h + 1036 \cdot 70k$	38·82		$941 \cdot 63h + 944 \cdot 28k$	38·76		
„ 18	11 30a.m.	$999 \cdot 83h + 1005 \cdot 38k$	40·00	−0·13	$926 \cdot 58h + 931 \cdot 88k$	39·80	−0·13	2
		$1000 \cdot 55h + 1004 \cdot 45k$	40·00		$925 \cdot 95h + 930 \cdot 50k$	39·79		
„ „	5 op.m.	$1005 \cdot 58h + 992 \cdot 28k$	40·28	−0·13	$902 \cdot 33h + 946 \cdot 88k$	40·06	−0·13	3
		$1006 \cdot 10h + 989 \cdot 40k$	40·27		$902 \cdot 83h + 945 \cdot 63k$	40·06		
„ 20	4 op.m.	$1225 \cdot 80h + 1228 \cdot 88k$	39·71	−0·13	$398 \cdot 45h + 384 \cdot 13k$	96·50	−0·20	4
		$1227 \cdot 15h + 1228 \cdot 23k$	39·69		$398 \cdot 80h + 380 \cdot 28k$	96·57		

Date.		Bronze Bar.			Steel Bar.			No. of Comparison.
Day.	Hour.	Micrometer Measurements.	Obs^d Temp.	Corr^n to Therm^r.	Micrometer Measurements.	Obs^d Temp.	Corr^n to Therm^r.	
1865.	H. M.		°			°		
Feb. 20	5 0p.m.	1419·58 *h* + 1407·90 *k* 1420·95 *h* + 1406·60 *k*	39·90 39·89	−0·13	579·05 *h* + 623·50 *k* 581·93 *h* + 622·75 *k*	95·01 95·01	−0·26	5
„ 21	3 15p.m.	1407·38 *h* + 1386·13 *k* 1409·10 *h* + 1386·43 *k*	40·55 40·54	−0·13	684·25 *h* + 690·68 *k* 686·60 *h* + 690·20 *k*	88·60 88·59	−0·27	6
„ „	4 7p.m.	1406·33 *h* + 1386·18 *k* 1408·10 *h* + 1385·75 *k*	40·69 40·69	−0·13	689·88 *h* + 689·88 *k* 690·05 *h* + 688·98 *k*	88·68 88·70	−0·27	7
„ „	4 55p.m.	1398·98 *h* + 1390·85 *k* 1400·20 *h* + 1388·65 *k*	40·80 40·80	−0·13	683·90 *h* + 693·90 *k* 686·23 *h* + 692·70 *k*	88·81 88·83	−0·27	8
„ 22	5 15p.m.	1393·48 *h* + 1392·43 *k* 1392·80 *h* + 1390·40 *k*	41·34 41·33	−0·14	719·35 *h* + 765·43 *k* 722·30 *h* + 764·43 *k*	85·40 85·43	−0·25	9
„ 24	4 17p.m.	1355·28 *h* + 1382·70 *k* 1355·70 *h* + 1380·45 *k*	42·96 42·95	−0·17	610·88 *h* + 634·05 *k* 612·40 *h* + 632·95 *k*	94·92 94·96	−0·25	10
„ „	5 20p.m.	1350·50 *h* + 1378·28 *k* 1355·28 *h* + 1379·35 *k*	43·14 43·15	−0·17	627·68 *h* + 655·08 *k* 627·18 *h* + 653·18 *k*	93·68 93·71	−0·25	11
„ 25	12 noon	1348·33 *h* + 1357·90 *k* 1351·50 *h* + 1359·25 *k*	43·87 43·86	−0·17	583·18 *h* + 587·00 *k* 583·43 *h* + 584·63 *k*	98·22 98·21	−0·16	12
„ „	0 50p.m.	1364·40 *h* + 1336·03 *k* 1365·60 *h* + 1335·68 *k*	44·00 43·99	−0·17	575·45 *h* + 571·15 *k* 575·03 *h* + 569·58 *k*	98·76 98·77	−0·16	13
„ „	3 47p.m.	1336·80 *h* + 1343·58 *k* 1337·90 *h* + 1342·65 *k*	44·38 44·41	−0·17	599·00 *h* + 603·93 *k* 598·75 *h* + 600·65 *k*	96·62 96·63	−0·20	14
„ „	4 37p.m.	1338·70 *h* + 1339·00 *k* 1339·88 *h* + 1337·00 *k*	44·55 44·56	−0·17	591·83 *h* + 617·78 *k* 593·80 *h* + 617·40 *k*	96·39 96·37	−0·20	15
„ 27	0 45p.m.	539·18 *h* + 537·95 *k* 540·40 *h* + 533·38 *k*	99·30 99·24	−0·14	1654·80 *h* + 1638·38 *k* 1656·00 *h* + 1637·45 *k*	42·95 42·96	−0·15	16
„ „	5 0p.m.	744·03 *h* + 763·85 *k* 745·63 *h* + 763·13 *k*	89·11 89·08	−0·32	1651·50 *h* + 1626·30 *k* 1653·33 *h* + 1624·78 *k*	43·81 43·81	−0·13	17
„ 28	11 42a.m.	608·98 *h* + 617·53 *k* 612·03 *h* + 616·58 *k*	95·89 95·90	−0·25	1617·95 *h* + 1632·70 *k* 1620·48 *h* + 1633·13 *k*	45·11 45·13	−0·13	18
„ „	4 0p.m.	668·70 *h* + 682·08 *k* 668·48 *h* + 679·30 *k*	93·32 93·29	−0·27	1605·10 *h* + 1637·03 *k* 1605·38 *h* + 1636·38 *k*	45·65 45·66	−0·17	19
„ „	4 35p.m.	667·83 *h* + 672·73 *k* 668·95 *h* + 672·35 *k*	93·60 93·63	−0·27	1615·58 *h* + 1625·80 *k* 1616·30 *h* + 1624·68 *k*	45·76 45·78	−0·17	20
Mar. 3	4 0p.m.	667·00 *h* + 683·85 *k* 669·55 *h* + 684·18 *k*	93·69 93·68	−0·27	1613·88 *h* + 1625·18 *k* 1614·78 *h* + 1623·90 *k*	46·41 46·41	−0·17	21
„ „	5 20p.m.	692·58 *h* + 692·75 *k* 696·38 *h* + 692·68 *k*	92·77 92·79	−0·26	1606·80 *h* + 1624·50 *k* 1608·88 *h* + 1623·33 *k*	46·77 46·78	−0·17	22
„ 4	11 10a.m.	1042·48 *h* + 1101·10 *k* 1041·60 *h* + 1097·08 *k*	74·53 74·52	−0·19	1611·23 *h* + 1624·75 *k* 1612·70 *h* + 1624·03 *k*	46·33 46·33	−0·17	23
„ „	12 noon	1058·43 *h* + 1068·53 *k* 1060·80 *h* + 1066·98 *k*	74·88 74·90	−0·19	1613·05 *h* + 1626·55 *k* 1612·83 *h* + 1624·05 *k*	46·32 46·35	−0·17	24
„ „	0 30p.m.	1054·93 *h* + 1062·68 *k* 1056·25 *h* + 1060·95 *k*	75·19 75·20	−0·19	1612·05 *h* + 1626·38 *k* 1613·13 *h* + 1625·20 *k*	46·35 46·37	−0·17	25

Date.		Bronze Bar.			Steel Bar.			No. of Comparison.
Day.	Hour.	Micrometer Measurements.	Obs^d Temp.	Corr^n to Therm^r.	Micrometer Measurements.	Obs^d Temp.	Corr^n to Therm^r.	
1865.	H. M.		°			°		
Mar. 4	3 45p.m.	724·25 *h* + 732·30 *k* 725·58 *h* + 730·30 *k*	74·40 74·42	−0·20	1269·85 *h* + 1271·90 *k* 1271·43 *h* + 1270·15 *k*	46·44 46·47	−0·17	26
,, ,,	4 24p.m.	722·80 *h* + 738·73 *k* 722·65 *h* + 735·35 *k*	74·30 74·31	−0·20	1267·23 *h* + 1276·30 *k* 1267·70 *h* + 1273·55 *k*	46·49 46·49	−0·17	27
,, 6	0 10p.m.	1011·18 *h* + 1007·33 *k* 1013·05 *h* + 1004·28 *k*	46·82 46·83	−0·19	1009·28 *h* + 1004·90 *k* 1012·60 *h* + 1003·63 *k*	44·83 44·84	−0·13	28
,, ,,	0 53p.m.	1006·18 *h* + 1014·23 *k* 1007·55 *h* + 1012·53 *k*	46·76 46·78	−0·19	1007·75 *h* + 1009·55 *k* 1007·75 *h* + 1006·55 *k*	44·83 44·84	−0·13	29
,, ,,	3 42p.m.	1015·10 *h* + 1016·75 *k* 1017·73 *h* + 1015·58 *k*	46·47 46·48	−0·19	1012·43 *h* + 1001·28 *k* 1015·73 *h* + 999·55 *k*	44·81 44·82	−0·13	30
,, ,,	4 15p.m.	1015·50 *h* + 1018·83 *k* 1016·53 *h* + 1017·80 *k*	46·45 46·45	−0·19	1017·10 *h* + 999·40 *k* 1018·50 *h* + 997·38 *k*	44·81 44·82	−0·13	31
,, ,,	4 45p.m.	1018·03 *h* + 1018·35 *k* 1019·08 *h* + 1016·80 *k*	46·42 46·42	−0·19	1007·63 *h* + 1007·13 *k* 1009·30 *h* + 1006·60 *k*	44·84 44·84	−0·13	32
,, 7	11 35a.m.	471·83 *h* + 472·08 *k* 472·68 *h* + 467·75 *k*	98·93 98·93	−0·15	1549·18 *h* + 1556·53 *k* 1551·28 *h* + 1555·33 *k*	44·34 44·35	−0·12	33
,, ,,	0 18p.m.	475·80 *h* + 459·50 *k* 478·53 *h* + 459·10 *k*	99·02 99·04	−0·13	1545·65 *h* + 1558·88 *k* 1547·13 *h* + 1558·75 *k*	44·40 44·42	−0·12	34
,, ,,	4 0p.m.	523·73 *h* + 524·45 *k* 523·73 *h* + 518·98 *k*	96·27 96·28	−0·25	1543·05 *h* + 1554·93 *k* 1544·18 *h* + 1552·75 *k*	44·76 44·78	−0·13	35
,, ,,	4 56p.m.	499·58 *h* + 505·83 *k* 500·63 *h* + 501·95 *k*	97·36 97·38	−0·19	1546·03 *h* + 1548·58 *k* 1548·40 *h* + 1547·43 *k*	44·89 44·89	−0·13	36
,, 8	0 30p.m.	600·28 *h* + 599·98 *k* 601·90 *h* + 598·58 *k*	81·50 81·51	−0·27	1306·15 *h* + 1318·23 *k* 1308·00 *h* + 1317·30 *k*	44·91 44·95	−0·13	37
,, ,,	4 5p.m.	609·55 *h* + 604·53 *k* 611·30 *h* + 602·25 *k*	81·20 81·20	−0·25	1310·63 *h* + 1313·50 *k* 1310·98 *h* + 1308·98 *k*	44·89 44·93	−0·13	38
,, ,,	4 50p.m.	601·43 *h* + 606·48 *k* 602·53 *h* + 603·93 *k*	81·31 81·29	−0·25	1309·55 *h* + 1313·25 *k* 1310·88 *h* + 1310·90 *k*	44·96 44·97	−0·13	39
,, 9	0 50p.m.	866·60 *h* + 897·25 *k* 864·65 *h* + 891·53 *k*	81·84 81·84	−0·25	1130·18 *h* + 1141·80 *k* 1132·68 *h* + 1140·00 *k*	80·08 80·09	−0·13	40
,, ,,	4 15p.m.	911·80 *h* + 921·30 *k* 914·23 *h* + 919·55 *k*	80·11 80·16	−0·22	1144·15 *h* + 1161·20 *k* 1145·80 *h* + 1160·55 *k*	78·65 78·65	−0·14	41
,, ,,	5 0p.m.	911·13 *h* + 910·23 *k* 912·43 *h* + 908·05 *k*	80·38 80·39	−0·22	1140·03 *h* + 1157·88 *k* 1141·78 *h* + 1156·85 *k*	79·02 79·01	−0·14	42
,, 10	0 12p.m.	922·83 *h* + 917·18 *k* 921·23 *h* + 913·03 *k*	91·72 91·70	−0·27	1228·28 *h* + 1237·23 *k* 1229·95 *h* + 1233·98 *k*	90·76 90·76	−0·28	43
,, ,,	4 7p.m.	982·58 *h* + 980·28 *k* 984·25 *h* + 978·28 *k*	88·74 88·74	−0·32	1249·90 *h* + 1277·85 *k* 1251·53 *h* + 1275·68 *k*	88·54 88·53	−0·27	44
,, ,,	4 55p.m.	990·63 *h* + 994·68 *k* 991·73 *h* + 992·93 *k*	88·12 88·14	−0·32	1261·23 *h* + 1271·65 *k* 1262·55 *h* + 1269·48 *k*	88·43 88·44	−0·27	45
,, 11	11 30a.m.	1039·53 *h* + 1034·65 *k* 1043·60 *h* + 1035·18 *k*	41·48 41·49	−0·14	954·53 *h* + 970·85 *k* 953·08 *h* + 965·75 *k*	42·28 42·27	−0·13	46

Date.		Bronze Bar.			Steel Bar.			No. of Comparison.
Day.	Hour.	Micrometer Measurements.	Obs^d Temp.	Corrⁿ to Therm^r.	Micrometer Measurements.	Obs^d Temp.	Corrⁿ to Therm^r.	
1865. Mar. 11	H. M. 0 12p.m.	1039·58h + 1043·43 k 1040·45h + 1041·63 k	° 41·36 41·38	−0·14	960·68h + 969·15 k 961·30h + 968·73 k	° 42·02 42·05	−0·13	47
„ „	0 40p.m.	1039·10h + 1045·90 k 1040·28h + 1043·58 k	41·33 41·34	−0·14	958·53h + 975·05 k 959·20h + 973·63 k	41·94 41·94	−0·13	48
„ „	4 5p.m.	1037·23h + 1038·28 k 1038·35h + 1037·15 k	41·49 41·50	−0·14	961·83h + 975·80 k 962·95h + 973·93 k	41·74 41·75	−0·13	49
„ „	4 35p.m.	1036·58h + 1038·03 k 1036·28h + 1035·78 k	41·57 41·60	−0·14	963·90h + 973·15 k 964·65h + 972·18 k	41·76 41·78	−0·13	50

The mean of the results by the two observers in each comparison being taken, and the temperatures corrected for errors of thermometers, we get the results shown in the next table, where the *order* of the comparisons has for convenience been altered.

No. of Comparison.	Difference of Length in Micrometer Divisions.	Difference of Length in Millionths of a Yard.	Temperature Bronze.	Temperature Steel.	Remarks.
1	81·21h + 89·16k	+ 135·83	° 38·69	° 38·67	Both bars cold.
2	73·92h + 73·72k	117·70	39·87	39·66	„ „
3	103·26h + 44·58k	117·75	40·14	39·93	„ „
28	1·18h + 1·54k	2·17	46·63	44·70	„ „
29	− 0·89h + 5·33k	3·53	46·58	44·70	„ „
30	2·34h + 15·75k	14·44	46·28	44·68	„ „
31	− 1·78h + 19·93k	14·50	46·26	44·68	„ „
32	10·09h + 10·71k	16·58	46·23	44·71	„ „
46	87·76h + 66·62k	123·04	41·34	42·14	„ „
47	79·03h + 73·59k	121·66	41·23	41·90	„ „
48	80·82h + 70·40k	120·53	41·19	41·81	„ „
49	75·40h + 62·85k	110·19	41·35	41·61	„ „
50	72·15h + 64·24k	+ 108·72	41·44	41·64	„ „
9	672·31h + 626·49k	+ 1035·31	41·19	85·16	Bronze cold. Steel hot.
6	722·81h + 695·84k	1130·88	40·41	88·32	„ „
7	717·25h + 696·54k	1127·02	40·56	88·42	„ „
8	714·52h + 696·45k	1124·77	40·67	88·55	„ „
11	725·46h + 724·69k	1156·03	42·97	93·44	„ „
10	743·85h + 748·08k	1189·35	42·78	94·69	„ „
5	839·78h + 784·12k	1294·46	39·76	94·75	„ „
15	746·47h + 720·41k	1169·33	44·38	96·18	„ „
4	827·85h + 846·35k	1334·67	39·57	96·33	„ „
14	738·47h + 740·83k	1179·28	44·22	96·42	„ „
12	766·61h + 772·76k	1227·17	43·69	98·05	„ „
13	789·76h + 765·49k	+ 1239·78	43·82	98·60	„ „
27	544·74h + 537·89k	− 863·04	74·10	46·32	Bronze hot. Steel cold.
26	545·72h + 539·73k	865·29	74·21	46·28	„ „
23	569·93h + 525·30k	873·03	74·33	46·16	„ „
24	553·32h + 557·54k	885·57	74·70	46·16	„ „
25	557·00h + 563·97k	893·63	75·00	46·19	„ „

No. of Comparison.	Difference of Length in Micrometer Divisions.	Difference of Length in Millionths of a Yard.	Temperature.		Remarks.
			Bronze.	Steel.	
38	$700{\cdot}38h + 707{\cdot}85k$	−1122·63	80°·95	44°·78	Bronze hot. Steel cold.
39	$708{\cdot}24h + 706{\cdot}87k$	1128·10	81·05	44·83	,, ,,
37	$705{\cdot}99h + 718{\cdot}49k$	1135·59	81·23	44·80	,, ,,
17	$907{\cdot}59h + 862{\cdot}05k$	1410·66	88·77	43·68	,, ,,
22	$913{\cdot}36h + 931{\cdot}20k$	1470·48	92·52	46·60	,, ,,
19	$936{\cdot}65h + 956{\cdot}02k$	1508·83	93·03	45·48	,, ,,
20	$947{\cdot}55h + 952{\cdot}70k$	1514·85	93·34	45·60	,, ,,
21	$946{\cdot}05h + 940{\cdot}52k$	1503·93	93·41	46·24	,, ,,
18	$1008{\cdot}71h + 1015{\cdot}86k$	1613·96	95·64	44·99	,, ,,
35	$1019{\cdot}89h + 1032{\cdot}12k$	1635·84	96·02	44·64	,, ,,
36	$1047{\cdot}11h + 1044{\cdot}12k$	1667·09	97·18	44·76	,, ,,
33	$1077{\cdot}97h + 1086{\cdot}01k$	1725·10	98·78	44·22	,, ,,
34	$1069{\cdot}22h + 1099{\cdot}52k$	1728·93	98·90	44·29	,, ,,
16	$1115{\cdot}61h + 1102{\cdot}25k$	−1768·02	99·13	42·80	,, ,,
40	$265{\cdot}80h + 246{\cdot}51k$	− 408·38	81·59	79·95	Both bars hot.
41	$231{\cdot}95h + 240{\cdot}45k$	376·60	79·91	78·51	,, ,,
42	$229{\cdot}13h + 248{\cdot}23k$	380·57	80·16	78·87	,, ,,
43	$307{\cdot}09h + 320{\cdot}50k$	500·32	91·44	90·48	,, ,,
44	$267{\cdot}30h + 297{\cdot}49k$	450·29	88·42	88·26	,, ,,
45	$270{\cdot}70h + 276{\cdot}76k$	− 436·43	87·81	88·16	,, ,,

Let now at the temperature of 40°, the Steel bar exceed the Bronze in length by a quantity z; also let

$$y = \text{expansion of Bronze bar for } 1^\circ$$
$$y' = \text{expansion of Steel bar} \quad ,,$$

then the difference of length when the Bronze has the temperature t and the Steel the temperature t' will be

$$z + (t' - 40)\,y' - (t - 40)\,y; \tag{16}$$

and if the actual observed difference of length under these circumstances be n, there results from this observation the equation

$$(t - 40)\,y - (t' - 40)\,y' - z + n = 0 \tag{17}$$

Proceeding thus with all the comparisons, we get the following system of equations:—

$$\begin{aligned}
-\ 1\cdot31\,y + 1\cdot33\,y' - z + 135\cdot83 &= 0 \qquad (18)\\
-\ 0\cdot13\,y + 0\cdot34\,y' - z + 117\cdot70 &= 0\\
+\ 0\cdot14\,y + 0\cdot07\,y' - z + 117\cdot75 &= 0\\
6\cdot63\,y - 4\cdot70\,y' - z + 2\cdot17 &= 0\\
6\cdot58\,y - 4\cdot70\,y' - z + 3\cdot53 &= 0\\
6\cdot28\,y - 4\cdot68\,y' - z + 14\cdot44 &= 0\\
6\cdot26\,y - 4.68\,y' - z + 14\cdot50 &= 0\\
6\cdot23\,y - 4\cdot71\,y' - z + 16\cdot58 &= 0\\
1\cdot34\,y - 2\cdot14\,y' - z + 123.04 &= 0\\
1\cdot23\,y - 1\cdot90\,y' - z + 121\cdot66 &= 0
\end{aligned}$$

$$1{\cdot}19\,y - 1{\cdot}81\,y' - z + 120{\cdot}53 = 0$$
$$1{\cdot}35\,y - 1{\cdot}61\,y' - z + 110{\cdot}19 = 0$$
$$1{\cdot}44\,y - 1{\cdot}64\,y' - z + 108{\cdot}72 = 0$$
$$1{\cdot}19\,y - 45{\cdot}16\,y' - z + 1035{\cdot}31 = 0$$
$$0{\cdot}41\,y - 48{\cdot}32\,y' - z + 1130{\cdot}88 = 0$$
$$0{\cdot}56\,y - 48{\cdot}42\,y' - z + 1127{\cdot}02 = 0$$
$$0{\cdot}67\,y - 48{\cdot}55\,y' - z + 1124{\cdot}77 = 0$$
$$2{\cdot}97\,y - 53{\cdot}44\,y' - z + 1156{\cdot}03 = 0$$
$$+\ 2{\cdot}78\,y - 54{\cdot}69\,y' - z + 1189{\cdot}35 = 0$$
$$-\ 0{\cdot}24\,y - 54{\cdot}75\,y' - z + 1294{\cdot}46 = 0$$
$$+\ 4{\cdot}38\,y - 56{\cdot}18\,y' - z + 1169{\cdot}33 = 0$$
$$-\ 0{\cdot}43\,y - 56{\cdot}33\,y' - z + 1334{\cdot}67 = 0$$
$$+\ 4{\cdot}22\,y - 56{\cdot}42\,y' - z + 1179{\cdot}28 = 0$$
$$3{\cdot}69\,y - 58{\cdot}05\,y' - z + 1227{\cdot}17 = 0$$
$$3{\cdot}82\,y - 58{\cdot}60\,y' - z + 1239{\cdot}78 = 0$$
$$34{\cdot}10\,y - 6{\cdot}32\,y' - z - 863{\cdot}04 = 0$$
$$34{\cdot}21\,y - 6{\cdot}28\,y' - z - 865{\cdot}29 = 0$$
$$34{\cdot}33\,y - 6{\cdot}16\,y' - z - 873{\cdot}03 = 0$$
$$34{\cdot}70\,y - 6{\cdot}16\,y' - z - 885{\cdot}57 = 0$$
$$35{\cdot}00\,y - 6{\cdot}19\,y' - z - 893{\cdot}63 = 0$$
$$40{\cdot}95\,y - 4{\cdot}78\,y' - z - 1122{\cdot}63 = 0$$
$$41{\cdot}05\,y - 4{\cdot}83\,y' - z - 1128{\cdot}10 = 0$$
$$41{\cdot}23\,y - 4{\cdot}80\,y' - z - 1135{\cdot}59 = 0$$
$$48{\cdot}77\,y - 3{\cdot}68\,y' - z - 1410{\cdot}66 = 0$$
$$52{\cdot}52\,y - 6{\cdot}60\,y' - z - 1470{\cdot}48 = 0$$
$$53{\cdot}03\,y - 5{\cdot}48\,y' - z - 1508{\cdot}83 = 0$$
$$53{\cdot}34\,y - 5{\cdot}60\,y' - z - 1514{\cdot}85 = 0$$
$$53{\cdot}41\,y - 6{\cdot}24\,y' - z - 1503{\cdot}93 = 0$$
$$55{\cdot}64\,y - 4{\cdot}99\,y' - z - 1613{\cdot}96 = 0$$
$$56{\cdot}02\,y - 4{\cdot}64\,y' - z - 1635{\cdot}84 = 0$$
$$57{\cdot}18\,y - 4{\cdot}76\,y' - z - 1667{\cdot}09 = 0$$
$$58{\cdot}78\,y - 4{\cdot}22\,y' - z - 1725{\cdot}10 = 0$$
$$58{\cdot}90\,y - 4{\cdot}29\,y' - z - 1728{\cdot}93 = 0$$
$$59{\cdot}13\,y - 2{\cdot}80\,y' - z - 1768{\cdot}02 = 0$$
$$41{\cdot}59\,y - 39{\cdot}95\,y' - z - 408{\cdot}38 = 0$$
$$39{\cdot}91\,y - 38{\cdot}51\,y' - z - 376{\cdot}60 = 0$$
$$40{\cdot}16\,y - 38{\cdot}87\,y' - z - 380{\cdot}57 = 0$$
$$51{\cdot}44\,y - 50{\cdot}48\,y' - z - 500{\cdot}32 = 0$$
$$48{\cdot}42\,y - 48{\cdot}26\,y' - z - 450{\cdot}29 = 0$$
$$+\ 47{\cdot}81\,y - 48{\cdot}16\,y' - z - 436{\cdot}43 = 0$$

Proceeding by the method of least squares, we should now form three equations from which those values of $z\ y\ y'$ would result which make the sum of the squares of the residual errors a minimum. But we must first consider a peculiarity of the equations which may render it desirable to treat them in a slightly different manner. Although the equations are all of the same form yet they are divided into four groups distinctive in character. In the first thirteen we have comparisons of two cold bars, in the remainder we have comparisons of a cold bar with a hot one, or of two hot bars, both of them situated in a cold room. We must therefore expect that the residual errors of the first thirteen equations will be very much smaller than those of the remaining equations. But we must not give less weight to these last equations because of their being liable to greater errors, for the errors are not wholly errors of observation. Here then we have an anomaly which leads us to pursue a course somewhat different from the ordinary one. The first group of equations, it will be seen, enable us to ascertain the difference in length of the two bars at a certain low temperature, as 43° or thereabouts. In fact if we know the *proportion* of y, y', we can ascertain the difference of length at a temperature which can also be found out: let τ be that temperature, and let the first group of equations be—

$$\begin{aligned}(t_1 - \tau)\, y - (t'_1 - \tau)\, y' - u + n_1 &= 0\\ (t_2 - \tau)\, y - (t'_2 - \tau)\, y' - u + n_2 &= 0\\ &\vdots\\ (t_{13} - \tau)\, y - (t'_{13} - \tau)\, y' - u + n_{13} &= 0\end{aligned}$$

Let the mean of the temperatures be $t_0\; t'_0$; and of the observed differences of length let n_0 be the mean: then $u = n_0$ if

$$(t_0 - \tau)\, y - (t'_0 - \tau)\, y' = 0;$$

or which is the same if

$$\tau = \tfrac{1}{2}\,(t_0 + t'_0) + \tfrac{1}{2}\,(t_0 - t'_0)\,\frac{y + y'}{y - y'}$$

Now an experimental calculation gives $y = 32 \cdot 8 \quad y' = 21 \cdot 1$; thus we get

$$\begin{aligned}\tau &= 43^\circ \cdot 75\\ u &= 77 \cdot 43\end{aligned}$$

that is at $43^\circ \cdot 75$ the difference of length is $77 \cdot 43$, the steel bar being the longer. We shall now refer all our other equations to the temperature of $43^\circ \cdot 75$, thus z is eliminated and the equations become as follow:—

$$\begin{aligned}
- 2\cdot56\, y - 41\cdot41\, y' + 957\cdot88 &= 0 \qquad (19)\\
- 3\cdot34\, y - 44\cdot57\, y' + 1053\cdot45 &= 0\\
- 3\cdot19\, y - 44\cdot67\, y' + 1049\cdot59 &= 0\\
- 3\cdot08\, y - 44\cdot80\, y' + 1047\cdot34 &= 0\\
- 0\cdot78\, y - 49\cdot69\, y' + 1078\cdot60 &= 0\\
- 0\cdot97\, y - 50\cdot94\, y' + 1111\cdot92 &= 0\\
- 3\cdot99\, y - 51\cdot00\, y' + 1217\cdot03 &= 0\\
+ 0\cdot63\, y - 52\cdot43\, y' + 1091\cdot90 &= 0\\
- 4\cdot18\, y - 52\cdot58\, y' + 1257\cdot24 &= 0\\
+ 0\cdot47\, y - 52\cdot67\, y' + 1101\cdot85 &= 0\\
- 0\cdot06\, y - 54\cdot30\, y' + 1149\cdot74 &= 0\\
+ 0\cdot07\, y - 54\cdot85\, y' + 1162\cdot35 &= 0
\end{aligned}$$

$$30{\cdot}35\,y - 2{\cdot}57\,y' - 940{\cdot}47 = 0$$
$$30{\cdot}46\,y - 2{\cdot}53\,y' - 942{\cdot}72 = 0$$
$$30{\cdot}58\,y - 2{\cdot}41\,y' - 950{\cdot}46 = 0$$
$$30{\cdot}95\,y - 2{\cdot}41\,y' - 963{\cdot}00 = 0$$
$$31{\cdot}25\,y - 2{\cdot}44\,y' - 971{\cdot}06 = 0$$
$$37{\cdot}20\,y - 1{\cdot}03\,y' - 1200{\cdot}06 = 0$$
$$37{\cdot}30\,y - 1{\cdot}08\,y' - 1205{\cdot}53 = 0$$
$$37{\cdot}48\,y - 1{\cdot}05\,y' - 1213{\cdot}02 = 0$$
$$45{\cdot}02\,y + 0{\cdot}07\,y' - 1488{\cdot}09 = 0$$
$$48{\cdot}77\,y - 2{\cdot}85\,y' - 1547{\cdot}91 = 0$$
$$49{\cdot}28\,y - 1{\cdot}73\,y' - 1586{\cdot}26 = 0$$
$$49{\cdot}59\,y - 1{\cdot}85\,y' - 1592{\cdot}28 = 0$$
$$49{\cdot}66\,y - 2{\cdot}49\,y' - 1581{\cdot}36 = 0.$$
$$51{\cdot}89\,y - 1{\cdot}24\,y' - 1691{\cdot}39 = 0$$
$$52{\cdot}27\,y - 0{\cdot}89\,y' - 1713{\cdot}27 = 0$$
$$53{\cdot}43\,y - 1{\cdot}01\,y' - 1744\ 52 = 0$$
$$55{\cdot}03\,y - 0{\cdot}47\,y' - 1802{\cdot}53 = 0$$
$$55{\cdot}15\,y - 0{\cdot}54\,y' - 1806{\cdot}36 - 0$$
$$55{\cdot}38\,y + 0{\cdot}95\,y' - 1845{\cdot}45 = 0$$
$$37{\cdot}84\,y - 36{\cdot}20\,y' - 485{\cdot}81 = 0$$
$$36{\cdot}16\,y - 34{\cdot}76\,y' - 454{\cdot}03 = 0$$
$$36{\cdot}41\,y - 35{\cdot}12\,y' - 458{\cdot}00 = 0$$
$$47{\cdot}69\,y - 46{\cdot}73\,y' - 577{\cdot}75 = 0$$
$$44{\cdot}67\,y - 44{\cdot}51\,y' - 527{\cdot}72 = 0$$
$$44{\cdot}06\,y - 44{\cdot}41\,y' - 513{\cdot}86 = 0$$

These equations represent comparisons taken under similar circumstances, they are consequently of the same degree of accuracy, and we shall solve them by the method of least squares. The final equations are

$$\begin{aligned} 48422{\cdot}85\,y - 10192{\cdot}41\,y' - 1379838{\cdot}86 &= 0 \\ -10192{\cdot}41\,y + 39561{\cdot}98\,y' - 502559{\cdot}64 &= 0 \end{aligned} \qquad (20)$$

Putting A, B, for the absolute terms of these equations, they become on elimination

$$\begin{aligned} y + {\cdot}000021835\ \mathrm{A} + {\cdot}000005625\ \mathrm{B} &= 0 \\ y' + {\cdot}000005625\ \mathrm{A} + {\cdot}000026726\ \mathrm{B} &= 0 \end{aligned} \qquad (21)$$

and restoring again the values of A, B, we get

$$\begin{aligned} y &= 32{\cdot}9566 : \textit{Reciprocal of weight} = {\cdot}00002183 \\ y' &= 21{\cdot}1938 : \qquad ,, \qquad ,, \qquad = {\cdot}00002673 \end{aligned} \qquad (22)$$

The errors of the different comparisons resulting from these values of y and y' together with the value of z or u at $43^\circ{\cdot}75$, are as shown in the following Table:—

Both cold.		Steel hot. Bronze cold.		Bronze hot. Steel cold.		Both hot.	
No. of Comp.	Error.	No. of Comp.	Error.	No. of Comp.	Error.	No. of Comp.	Error.
1	−0·70	9	−4·12	27	+5·29	40	−5·95
2	−0·92	6	−1·23	26	+7·52	41	+0·98
3	+2·31	7	−2·27	23	+6·28	42	−2·38
28	−0·48	8	−3·55	24	+5·93	43	+3·56
29	−0·77	11	−0·23	25	+7·12	44	+1·11
30	+0·68	10	+0·34	38	+4·10	45	−3·01
31	+0·08	5	+4·65	39	+0·86		
32	+0·54	15	+1·47	37	−0·06		
46	+0·31	4	+5·11	17	−2·90		
47	+0·39	14	+1·06	22	−1·02		
48	−0·15	12	−3·05	19	+1·18		
49	−0·98	13	+2·18	20	+2·83		
50	−0·12			21	+2·50		
				18	−7·55		
				35	−9·49		
				36	−5·06		
				33	+1·11		
				34	−0·24		
				16	−0·18		

Here we see that the errors of the comparisons when the bars are both cold are very much smaller than when one or both of the bars are hot. Of the thirteen errors in the first group six are greater than 0·54, six less, and one equal to that quantity. Hence we infer that the probable error of u is about $\pm 0 \cdot 15$. Now the absolute terms of the equations from which we have determined y, y', are all dependent on the adopted value of u, so that the errors in the three last columns of the above Table are also affected by the probable error of u. But the errors in these columns are so very large in comparison with the probable error of u that we may lay aside this consideration.

The sum of the squares of the errors of the 37 equations is 592·95, hence the probable error of a comparison, one or both bars being hot, is

$$\cdot 674 \sqrt{\frac{592 \cdot 95}{37-2}} = \pm\, 2{\cdot}77 \qquad (23)$$

It is not easy to account for the large errors. The heated air rising from the hot bar just under the observing microscopes often caused an irregular refraction and apparent motion in the object viewed, similar to that observed in looking with a telescope at a terrestrial object on a hot day. This was sometimes troublesome, and may have produced an error of as much as two micrometer divisions at times, but not probably more. Also the exceeding freedom of the bars from being suspended, was another cause of a small amount of error, as carriages passing in the neighbourhood produced an oscillation of sometimes as much as $\pm$ 1·5 division. But this hardly affected the certainty of the bisections at any time.

Another, probably more serious, source of error was an irregularity in the distance of the copper tanks from the bar at different parts of its length. The average distance was about $\frac{3}{16}$ of an inch, but the difficulty of making copper tanks ten feet long perfectly straight involved some variations on that distance. In order to remedy this, after the conclusion of the observations just recorded, the boxes were taken to pieces and the tanks somewhat improved as to straightness. The tanks were then replaced at a greater distance than before, namely, about $\frac{7}{16}$ inch.

The probable errors of y, y', are,—

$$y \ldots\ldots\ldots \pm 2\cdot 77 \sqrt{\cdot 00002183} = \pm \cdot 0129 \qquad (24)$$
$$y' \ldots\ldots\ldots \pm 2\cdot 77 \sqrt{\cdot 00002673} = \pm \cdot 0143$$

so that the final results for expansion are,—

$$\textit{Expansion of Bronze Bar} = 32\cdot 9566 \pm \cdot 0129 \qquad (25)$$
$$\textit{Expansion of Steel Bar} = 21\cdot 1938 \pm \cdot 0143$$

If we express these quantities as a fraction of the bar's length we get the ordinary form of " co-efficient of expansion " thus,—

$$\textit{Co-efficient of Expansion of Bronze} = \cdot 0000098870 \pm \cdot 0000000039$$
$$\textit{Co-efficient of Expansion of Steel} = \cdot 0000063581 \pm \cdot 0000000043$$

5.

In the second series of experiments which extended over 12 days, viz., from the 29th April to the 10th May, the roller frames were restricted to a strictly vertical movement, so as to obtain the co-efficient of expansion under circumstances more similar to those in which the bars will be actually used. The bars then, it is to be understood, in the second series of experiments were not swinging, but resting on two rollers, which admitted only of adjustment for level and focus.

The observations are given in the annexed Table :—

Date.		Bronze Bar.			Steel Bar.			No. of Comparison.
Day.	Hour.	Micrometer Measurements.	Obs^d Temp.	Corr^n to Therm^r.	Micrometer Measurements.	Obs^d Temp.	Corr^n to Therm^r.	
1865.	H. M.		°	°		°	°	
April 29	11 50a.m.	1040·85 h + 1045·43 k	57·30	−0·09	858·43 h + 863·73 k	74·89	−0·26	1
		1042·20 h + 1044·55 k	57·31		858·85 h + 863·40 k	74·91		
„ „	0 30p.m.	1042·33 h + 1044·25 k	57·30	−0·09	855·78 h + 870·05 k	74·81	−0·26	2
		1043·25 h + 1044·20 k	57·29		856·00 h + 868·90 k	74·80		
„ „	3 35p.m.	1041·83 h + 1044·93 k	57·27	−0·09	865·78 h + 850·68 k	75·04	−0·26	3
		1042·55 h + 1043·30 k	57·26		868·03 h + 851·18 k	75·05		
„ „	4 20p.m.	1042·25 h + 1046·60 k	57·28	−0·09	860·80 h + 856·60 k	75·01	−0·26	4
		1043·08 h + 1046·08 k	57·27		861·90 h + 856·85 k	75·00		
May 1		1092·10 h + 1091·48 k	54·33	−0·14	706·50 h + 718·90 k	85·05	−0.30	5
		1092·50 h + 1089·23 k	54·32		708·13 h + 719·40 k	85·05		
„ „	4 0p.m.	1091·08 h + 1089·05 k	54·41	−0·14	706·93 h + 711·18 k	85·28	−0·30	6
		1092·40 h + 1087·88 k	54·40		708·58 h + 710·38 k	85·28		
„ 3	11 35a.m.	1314·25 h + 1320·23 k	56·16	−0·10	821·18 h + 831·83 k	96·34	−0·27	7
		1315·00 h + 1318·03 k	56·17		822·08 h + 830·25 k	96·34		
„ „	0 5p.m.	1313·68 h + 1318·13 k	56·25	−0·10	816·05 h + 837·28 k	96·24	−0·27	8
		1316·38 h + 1317·90 k	56·24		819·20 h + 838·28 k	96·22		

Date.		Bronze Bar.			Steel Bar.			No. of Comparison.
Day.	Hour.	Micrometer Measurements.	Obs[d] Temp.	Corr[n] to Therm[r].	Micrometer Measurements.	Obs[d] Temp.	Corr[n] to Therm[r].	
1865.	H. M.		°	°		°	°	
May 3	3 35p.m.	1305·73 *h* + 1313·90 *k* 1306·55 *h* + 1312·73 *k*	56·59 56·61	−0·09	824·75 *h* + 831·08 *k* 826·10 *h* + 830·55 *k*	96·32 96·33	−0·27	9
„ „	4 8p.m.	1303·10 *h* + 1314·00 *k* 1304·88 *h* + 1313·30 *k*	56·69 56·70	−0·09	823·60 *h* + 837·43 *k* 823·90 *h* + 834·05 *k*	96·21 96·21	−0·27	10
„ 4	11 30a.m.	919·23 *h* + 920·50 *k* 919·58 *h* + 918·25 *k*	75·96 75·95	−0·19	1298·03 *h* + 1298·38 *k* 1299·00 *h* + 1297·45 *k*	61·04 61·05	−0·19	11
„ „	12 0 noon	923·23 *h* + 898·43 *k* 929·28 *h* + 901·45 *k*	76·20 76·21	−0·19	1298·23 *h* + 1299·63 *k* 1300·03 *h* + 1299·90 *k*	60·95 60·94	−0·19	12
„ 5	11 10a.m.	923·80 *h* + 917·73 *k* 925·75 *h* + 916·50 *k*	76·03 76·02	−0·19	1340·55 *h* + 1341·85 *k* 1342·95 *h* + 1340·98 *k*	58·03 58·03	−0·19	13
„ „	11 50a.m.	915·50 *h* + 918·50 *k* 915·98 *h* + 915·28 *k*	76·28 76·28	−0·19	1339·60 *h* + 1344·73 *k* 1341·95 *h* + 1343·88 *k*	58·04 58·03	−0·19	14
„ „	3 45p.m.	816·10 *h* + 809·28 *k* 819·75 *h* + 809·93 *k*	81·38 81·36	−0·27	1336·30 *h* + 1345·65 *k* 1337·98 *h* + 1344·28 *k*	58·07 58·08	−0·19	15
„ „	4 20p.m.	810·98 *h* + 818·85 *k* 811·70 *h* + 815·83 *k*	81·34 81·34	−0·27	1338·93 *h* + 1344·50 *k* 1340·85 *h* + 1343·83 *k*	58·12 58·12	−0·19	16
„ 6	11 7a.m.	492·33 *h* + 517·15 *k* 491·58 *h* + 513·85 *k*	96·44 96·42	−0·27	1348·25 *h* + 1350·08 *k* 1350·38 *h* + 1349·23 *k*	57·57 57·58	−0·18	17
„ „	11 48a.m.	504·93 *h* + 492·88 *k* 511·70 *h* + 495·93 *k*	96·47 96·45	−0·27	1348·60 *h* + 1349·65 *k* 1350·70 *h* + 1349·28 *k*	57·65 57·67	−0·18	18
„ „	3 40p.m.	532·65 *h* + 531·13 *k* 533·35 *h* + 530·05 *k*	95·06 95·08	−0·28	1349·90 *h* + 1345·25 *k* 1350·95 *h* + 1344·28 *k*	57·95 57·95	−0·19	19
„ „	4 15p.m.	532·75 *h* + 526·93 *k* 533·73 *h* + 524·68 *k*	95·20 95·20	−0·27	1347·15 *h* + 1347·30 *k* 1347·93 *h* + 1345·20 *k*	58·01 58·00	−0·19	20
„ 8	4 45p.m.	1302·35 *h* + 1331·83 *k* 1299·50 *h* + 1324·53 *k*	56·73 56·73	−0·09	1362·75 *h* + 1367·83 *k* 1365·60 *h* + 1366·50 *k*	56·54 56·54	−0·16	21*
„ 9	10 45a.m.	1318·45 *h* + 1327·48 *k* 1319·48 *h* + 1325·23 *k*	56·52 56·54	−0·09	1364·28 *h* + 1365·55 *k* 1369·50 *h* + 1367·43 *k*	56·46 56·47	−0·16	22
„ „	11 15a.m.	1320·23 *h* + 1323·40 *k* 1321·65 *h* + 1322·65 *k*	56·53 56·54	−0·09	1368·85 *h* + 1364·75 *k* 1371·55 *h* + 1364·10 *k*	56·48 56·49	−0·16	23
„ „	3 50p.m.	1322·50 *h* + 1319·88 *k* 1323·73 *h* + 1318·68 *k*	56·63 56·66	−0·09	1361·00 *h* + 1373·40 *k* 1363·25 *h* + 1372·45 *k*	56·54 56·54	−0·16	24
„ „	4 20p.m.	1319·50 *h* + 1326·80 *k* 1318·90 *h* + 1321·95 *k*	56·67 56·68	−0·09	1365·75 *h* + 1369·35 *k* 1367·80 *h* + 1367·30 *k*	56·55 56·57	−0·16	25
„ 10	10 30a.m.	1319·13 *h* + 1318·40 *k* 1321·30 *h* + 1316·88 *k*	56·86 56·85	−0·09	1363·95 *h* + 1362·45 *k* 1368·95 *h* + 1366·15 *k*	56·83 56·80	−0·16	26
„ „	11 0a.m.	1317·50 *h* + 1320·25 *k* 1318·80 *h* + 1319·53 *k*	56·87 56·85	−0·09	1371·93 *h* + 1361·50 *k* 1370·65 *h* + 1357·05 *k*	56·85 56·82	−0·16	27

* There are some apparent mistakes in the registry or reading of one of the micrometers on the Bronze Bar in this comparison which cannot be unravelled and therefore not safely corrected. The comparison is rejected.

Taking now the means of the results for the two observers in each comparison, and correcting the observed temperatures, we form the next table.

No. of Comparison.	Difference of Length in Micrometer Divisions.	Difference of Length in Millionths of a Yard.	Temperature. Bronze.	Temperature. Steel.	Remarks.
			°	°	
1	182·88 *h* + 181·43 *k*	+ 290·42	57·21	74·64	Bronze cold. Steel hot.
2	186·90 *h* + 174·75 *k*	288·28	57·20	74·54	" "
3	175·30 *h* + 193·18 *k*	293·77	57·17	74·78	" "
4	181·31 *h* + 189·62 *k*	295·71	57·18	74·74	" "
5	384·99 *h* + 371·20 *k*	602·80	54·18	84·75	" "
6	383·99 *h* + 377·68 *k*	607·18	54·26	84·98	" "
7	492·99 *h* + 488·09 *k*	782·09	56·06	96·07	" "
8	497·41 *h* + 480·23 *k*	779·33	56·14	95·96	" "
9	480·72 *h* + 482·49 *k*	768·15	56·51	96·05	" "
10	480·24 *h* + 477·91 *k*	+ 763·58	56·60	95·94	" "
11	379·11 *h* + 378·54 *k*	− 603·98	75·76	60·85	Bronze hot. Steel cold.
12	372·88 *h* + 399·82 *k*	616·02	76·01	60·75	" "
13	416·98 *h* + 424·30 *k*	670·67	75·83	57·84	" "
14	425·03 *h* + 427·41 *k*	679·55	76·09	57·84	" "
15	519·22 *h* + 535·36 *k*	840·72	81·10	57·89	" "
16	528·55 *h* + 526·82 *k*	841·32	81·07	57·93	" "
17	857·36 *h* + 834·15 *k*	1348·41	96·16	57·39	" "
18	841·34 *h* + 855·06 *k*	1352·36	96·19	57·48	" "
19	817·42 *h* + 814·17 *k*	1300·67	94·79	57·76	" "
20	814·30 *h* + 820·45 *k*	−1303·20	94·93	57·81	" "
22	47·93 *h* + 40·14 *k*	− 70·20	56·44	56·30	Both bars cold.
23	49·26 *h* + 41·40 *k*	72·26	56·44	56·32	"
24	39·01 *h* + 53·64 *k*	73·88	56·55	56·38	"
25	47·57 *h* + 43·95 *k*	72·96	56·58	56·40	"
26	46·24 *h* + 46·66 *k*	74·06	56·76	56·65	"
27	53·14 *h* + 39·38 *k*	− 73·74	56·77	56·67	"

In the last group of comparisons the mean of the temperatures of the Bronze Bar is $56°\cdot590$, and of the Steel $56°\cdot453$, whence, proceeding as before, we get $\tau = 56°\cdot84$, and u the mean of the corresponding measured differences of length $= -72\cdot85$; so that at $56°\cdot84$ the Steel bar is shorter than the Bronze by $72\cdot85$. Our equations of condition will then be of the form

$$y\,(t - 56°\cdot84) - y'\,(t' - 56°\cdot84) + n + 72\cdot85 = 0; \tag{26}$$

they will be found as follow:—

$$\begin{aligned}
0\cdot37\,y - 17\cdot80\,y' + 363\cdot27 &= 0 \\
0\cdot36\,y - 17\cdot70\,y' + 361\cdot13 &= 0 \\
0\cdot33\,y - 17\cdot94\,y' + 366\cdot62 &= 0 \\
0\cdot34\,y - 17\cdot90\,y' + 368\cdot55 &= 0 \\
-\,2\cdot66\,y - 27\cdot91\,y' + 675\cdot65 &= 0 \\
-\,2\cdot58\,y - 28\cdot14\,y' + 680\cdot03 &= 0 \\
-\,0\cdot78\,y - 39\cdot23\,y' + 854\cdot94 &= 0 \\
-\,0\cdot70\,y - 39\cdot12\,y' + 852\cdot18 &= 0 \\
-\,0\cdot33\,y - 39\cdot21\,y' + 841\cdot00 &= 0 \\
-\,0\cdot24\,y - 39\cdot10\,y' + 836\cdot43 &= 0
\end{aligned} \tag{27}$$

$$+18\cdot92\,y - 4\cdot01\,y' - 531\cdot13 = 0$$
$$+19\cdot17\,y - 3\cdot91\,y' - 543\cdot17 = 0$$
$$+18\cdot99\,y - 1\cdot00\,y' - 597\cdot82 = 0$$
$$+19\cdot25\,y - 1\cdot00\,y' - 606\cdot70 = 0$$
$$+24\cdot26\,y - 1\cdot05\,y' - 767\cdot87 = 0$$
$$+24\cdot23\,y - 1\cdot09\,y' - 768\cdot47 = 0$$
$$+39\cdot32\,y - 0\cdot55\,y' - 1275\cdot56 = 0$$
$$+39\cdot35\,y - 0\cdot64\,y' - 1279\cdot51 = 0$$
$$+37\cdot95\,y - 0\cdot92\,y' - 1227\cdot82 = 0$$
$$+38\cdot09\,y - 0\cdot97\,y' - 1230\cdot35 = 0$$

These solved by the method of least squares give

$$8633\cdot30\,y - 157\cdot42\,y' - 279487\cdot85 = 0 \qquad (28)$$
$$-157\cdot42\,y + 9016\cdot96\,y' - 185636\cdot93 = 0$$

writing A and B for the absolute terms of these equations, they become on eliminating $y\ y'$

$$y + \cdot00011587\,A + \cdot00000202\,B = 0 \qquad (29)$$
$$y' + \cdot00000202\,A + \cdot00011094\,B = 0$$

restoring the values of A and B, we get these results:—

$y = 32\cdot7591$ . . . . . . *Reciprocal of weight* $= \cdot0001159$ (30)

$y' = 21\cdot1594$ . . . . . . " " $= \cdot0001109$

The corresponding system of errors of comparisons is shown in the following Table:—

Both Bars cold.		Steel hot. Bronze cold.		Steel cold. Bronze hot.	
No. of Comp.	Error.	No. of Comp.	Error.	No. of Comp.	Error.
22	+0·98	1	−1·23	11	+3·82
23	−1·51	2	−1·60	12	+2·09
24	−0·80	3	−2·17	13	+3·11
25	+0·68	4	+0·94	14	+2·75
26	+0·19	5	−2·03	15	+4·65
27	+0·42	6	+0·08	16	+2·22
		7	−0·69	17	+0·89
		8	+1·49	18	−3·98
		9	+0·53	19	−4·08
		10	+1·24	20	−3·17

The sum of the squares of the errors of the 20 equations, (shown in the two right-hand columns,) is 124·38, hence the probable error of one comparison is,—

$$\pm\,\cdot674\sqrt{\frac{124\cdot38}{20-2}} = \pm\,1\cdot772 \qquad (31)$$

The probable errors of $y\ y'$ are consequently

y . . . . $\pm\,1\cdot772\,\sqrt{\cdot0001159} = \pm\,0\cdot0191$

y' . . . . $\pm\,1\cdot772\,\sqrt{\cdot0001109} = \pm\,0\cdot0187$

so that the final results for expansion are—

$$\textit{Expansion of Bronze Bar} = 32{\cdot}7591 \pm 0{\cdot}0191 \qquad (33)$$
$$\text{,, } \textit{Steel Bar} = 21{\cdot}1594 \pm 0{\cdot}0187$$

or in another form—

$$\textit{Co-efficient of Expansion of Bronze} = {\cdot}0000098277 \pm {\cdot}0000000057 \qquad (34)$$
$$\text{,, ,, } \textit{Steel} = {\cdot}0000063478 \pm {\cdot}0000000056$$

6.

The results to which we are conducted by the second series of experiments for expansion, show in the case of either bar a smaller co-efficient. In the case of the Steel bar the difference is immaterial, but in the case of the Bronze bar it is very sensible. The question now arises, is this difference due to the circumstance that the bars were not so free to expand as in the first experiments? If so, expansion must always be variable according to amount of friction of the rollers. An accumulation of oil and dust about the axles in course of time would produce a gradually diminishing rate of expansion.

The errors in the second series are materially smaller than in the first. This is, doubtless, owing in part to the increased distance of the tanks from the bars as before explained, and in part to the fact that the range of temperature was smaller. But there is this analogy between the two systems of residual errors, that in each case the errors are larger when the Bronze Bar is hot. In the first series, the average of the errors of comparison, when the Steel Bar is hot is 2·44, and when the Bronze is hot 3·75; in the second series the corresponding averages are 1·20, 3·07. This leads us to the conclusion that the computed probable error for the Bronze Bar is in either series *too small.*

Again, examining the errors of the comparisons with Bronze hot in either series, there is an inclination to a predominance of + errors at the lower temperatures, and − errors at the higher temperatures. This may or may not be accidental, it may arise from a rate of expansion varying with the temperature. It may be interesting to see the individual rates of expansion of the Bronze Bar that would suit the different comparisons with Bronze hot and Steel cold. They are shown in the following Table:—

Co-efficients of Expansion of Bronze Bar.

First Series of Experiments.

No. of Comparison.	Temperature.	Co-efficient of Expansion.	No. of Comparison.	Temperature.	Co-efficient of Expansion.
27	74°·1	·000009835	19	93°·0	·000009880
26	74·2	·000009813	20	93·3	·000009870
23	74·3	·000009825	21	93·4	·000009872
24	74·7	·000009829	18	95·6	·000009931
25	75·0	·000009819	35	96·0	·000009941
38	80·9	·000009854	36	97·2	·000009915
39	81·0	·000009880	33	98·8	·000009881
37	81·2	·000009887	34	98·9	·000009888
17	88·8	·000009906	16	99·1	·000009888
22	92·5	·000009893			

Temperature of the cold bar about 44°.

SECOND SERIES OF EXPERIMENTS.

No. of Comparison.	Temperature.	Co-efficient of Expansion.	No. of Comparison.	Temperature.	Co-efficient of Expansion.
11	75°·8	·000009767	16	81°·0	·000009800
12	76·0	·000009795	17	96·2	·000009821
13	75·8	·000009779	18	96·2	·000009858
14	76·1	·000009785	19	94·8	·000009860
15	81·1	·000009770	20	94·9	·000009852

Temperature of the cold bar about 57°.

Similarly we may exhibit the co-efficients of expansion of the Steel bar, which would satisfy the equations of conditions, or comparisons, when the Bronze is cold and Steel hot; They are as in the following Tables:—

CO-EFFICIENTS OF EXPANSION OF STEEL BAR.

FIRST SERIES OF EXPERIMENTS.

No. of Comparison.	Temperature.	Co-efficient of Expansion.	No. of Comparison.	Temperature.	Co-efficient of Expansion.
9	85°·2	·000006328	5	94°·7	·000006385
6	88·3	·000006350	15	96·2	·000006367
7	88·4	·000006343	4	96·3	·000006387
8	88·6	·000006334	14	96·4	·000006364
11	93·4	·000006357	12	98·0	·000006341
10	94·7	·000006360	3	98·6	·000006370

Cold bar about 42°.

SECOND SERIES OF EXPERIMENTS.

No. of Comparison.	Temperature.	Co-efficient of Expansion.	No. of Comparison.	Temperature.	Co-efficient of Expansion.
1	74°·6	·000006327	6	85°·0	·000006349
2	74·5	·000006321	7	96·1	·000006343
3	74·8	·000006312	8	96·0	·000006359
4	74·7	·000006363	9	96·0	·000006352
5	84·7	·000006326	10	95·9	·000006357

Cold bar about 56°.

It seems almost unnecessary to remark that final values for expansion are not to be gathered from these Tables, which are merely illustrative of the degree of precision attained. The final results remain as already stated, equations (25, 33).

7.

There remain a few comparisons with the Bronze bar cold and the Steel hot to be considered. In point of time these belong to the first series of experiments. They are as in the following Table:—

Date.		Bronze Bar.			Steel Bar.			No. of Comparison.
Day.	Hour.	Micrometer Measurements.	Obs^{d} Temp.	Corr^{n} to Therm^{r}.	Micrometer Measurements.	Obs^{d} Temp.	Corr^{n} to Therm^{r}.	
1865.	H. M.		°			°		
Mar. 15	10 50a.m.	1387·30 *h* + 1369·80 *k*	43·39	−0·17	1076·75 *h* + 1071·28 *k*	62·67	−0·13	51
		1388·70 *h* + 1369·00 *k*	43·38		1078·50 *h* + 1069·45	62·69		
„ „	11 25a.m.	1376·65 *h* + 1380·33 *k*	43·38	−0·17	1084·08 *h* + 1083·90 *k*	61·90	−0·13	52
		1380·50 *h* + 1378·05 *k*	43·37		1085·45 *h* + 1084·30 *k*	61·90		
„ 16	11 30a.m.	1386·18 *h* + 1389·85 *k*	42·86	−0·17	1185·48 *h* + 1190·85 *k*	53·89	−0·13	53
		1386·90 *h* + 1388·75 *k*	42·86		1186·15 *h* + 1189·83 *k*	53·89		
„ „	0 7p.m.	1388·15 *h* + 1389·00 *k*	42·87	−0·17	1191·68 *h* + 1194·38 *k*	53·56	−0·14	54
		1388·43 *h* + 1388·48 *k*	42·86		1192·18 *h* + 1193·88 *k*	53·57		
„ „	4 6p.m.	1387·50 *h* + 1387·43 *k*	42·89	−0·16	1208·08 *h* + 1231·90 *k*	51·35	−0·11	55
		1388·75 *h* + 1386·18 *k*	42·89		1212·88 *h* + 1233·43 *k*	51·36		
„ „	4 50p.m.	1388·58 *h* + 1385·03 *k*	42·94	−0·16	1226·30 *h* + 1224·80 *k*	51·09	−0·11	56
		1389·98 *h* + 1383·80 *k*	42·93		1227·48 *h* + 1224·13 *k*	51·10		
„ 17	0 10p.m.	1384·28 *h* + 1386·30 *k*	42·93	−0·16	1038·68 *h* + 1054·60 *k*	64·68	−0·18	57
		1385·83 *h* + 1385·40 *k*	42·93		1039·25 *h* + 1051·58 *k*	64·70		
„ „	0 50p.m.	1383·40 *h* + 1386·73 *k*	42·96	−0·16	1048·45 *h* + 1050·00 *k*	64·44	−0·18	58
		1385·23 *h* + 1385·88 *k*	42·97		1050·88 *h* + 1049·50 *k*	64·46		

Taking first the means of the pairs of results obtained by the two observers, then the difference of these means; and correcting the thermometer readings, we get the following:—

No. of Comparison.	Difference of Length in Micrometer Divisions.	Difference of Length in Millionths of a Yard.	Temperature.		Remarks.	
			Bronze.	Steel.		
			°	°		
51	310·27 *h* + 299·03 *k*	+ 485·71	43·22	62·55	Bronze cold.	Steel hot.
52	293·81 *h* + 295·09 *k*	469·46	43·21	61·77	„	„
53	200·72 *h* + 198·96 *k*	318·61	42·69	53·76	„	„
54	196·36 *h* + 194·61 *k*	311·67	42·70	53·43	„	„
55	177·65 *h* + 154·14 *k*	264·46	42·73	51·25	„	„
56	162·39 *h* + 159·95 *k*	256·96	42·78	50·99	„	„
57	346·09 *h* + 332·76 *k*	541·15	42·77	64·51	„	„
58	334·65 *h* + 336·56 *k*	+ 535·08	42·80	64·27	„	„

The equations of condition formed in the same manner as equations (19), namely, by the formula

$$(t - 43{\cdot}75)\,y - (t' - 43{\cdot}75)\,y' + n - 77{\cdot}43 = 0$$

are these—

$$
\begin{aligned}
-0\cdot53\,y - 18\cdot80\,y' + 408\cdot28 &= 0 \qquad (35)\\
-0\cdot54\,y - 18\cdot02\,y' + 392\cdot03 &= 0\\
-1\cdot06\,y - 10\cdot01\,y' + 241\cdot18 &= 0\\
-1\cdot05\,y - 9\cdot68\,y' + 234\cdot24 &= 0\\
-1\cdot02\,y - 7\cdot50\,y' + 187\cdot03 &= 0\\
-0\cdot97\,y - 7\cdot24\,y' + 179\cdot53 &= 0\\
-0\cdot98\,y - 20\cdot76\,y' + 463\cdot72 &= 0\\
-0\cdot95\,y - 20\cdot52\,y' + 457\cdot65 &= 0
\end{aligned}
$$

If here for y and y' we substitute the final values obtained from the first series, namely, $y = 32\cdot9566$ and $y' = 21\cdot1938$; the equations will be very far from satisfied, the residual errors being—

$$
\begin{aligned}
&-7\cdot63\\
&-7\cdot68\\
&-5\cdot91\\
&-5\cdot53\\
&-5\cdot54\\
&-5\cdot88\\
&-8\cdot56\\
&-8\cdot56
\end{aligned}
$$

Now this makes it clear that an entirely different value of y' is required in these comparisons. In order to obtain this value, substitute in the equations (35) $y = 32\cdot9566$ and they become—

$$
\begin{aligned}
-18\cdot80\,y' + 390\cdot81 &= 0\\
-18\cdot02\,y' + 374\cdot23 &= 0\\
-10\cdot01\,y' + 206\cdot24 &= 0\\
-9\cdot68\,y' + 199\cdot63 &= 0\\
-7\cdot50\,y' + 153\cdot41 &= 0\\
-7\cdot24\,y' + 147\cdot56 &= 0\\
-20\cdot76\,y' + 431\cdot42 &= 0\\
-20\cdot52\,y' + 426\cdot34 &= 0
\end{aligned}
$$

whence—

$$
\begin{aligned}
1832\cdot7785\,y' - 38011\cdot4188 &= 0\\
\therefore\ y' &= 20\cdot7398
\end{aligned}
$$

leaving the following errors of comparisons :—

$$
\begin{aligned}
&-0\cdot90\\
&-0\cdot50\\
&+1\cdot37\\
&+1\cdot13\\
&+2\cdot14\\
&+2\cdot60\\
&-0\cdot86\\
&-0\cdot76
\end{aligned}
$$

with the probable error of the determination of y' about $\pm\ \cdot024$.

The change of rate of expansion here brought out in the Steel bar from 21·1938 to 20·7398 is, doubtless, owing to the circumstance that in these last comparisons the temperature of the bar was not *sustained*. No current of warm water was running through the tanks, and, consequently, the bar was steadily, though very slowly, cooling. At 11 A.M. on 15th March the temperature of the bar was about 62°, and by 5 P.M. on the 16th it had cooled to 51°. On the morning of the 17th it was again heated to above 65° and allowed to cool.

This result would lead us to expect that if the rate of expansion of a bar be deduced from observations made upon that bar while in a state of cooling, though only slowly, such deduced expansion is probably smaller than the true value.

The total number of micrometer and thermometer readings in these expansion experiments, not including any of the observations for determination of the errors of thermometers, is 2720 micrometer, 2720 thermometer readings.

XVII.

RE-DETERMINATION OF THE EXPANSIONS

OF

OI_1 AND OI_2

1.

In a preceding section we have given the results of the experiments to which these bars were subjected in November 1857. The method adopted is open to objection on account of the inevitable state of cooling of the hot bar, tending to produce a constant error, and from the results of the last section it would seem probable that the expansions then obtained are too small. It was therefore thought desirable to obtain the expansions from the new apparatus.

The experiments or comparisons of the bars, hot and cold, extend over eleven days, between the 3rd and 16th of November 1865, in all, 38 comparisons. In the last two, OI_1 being cold and OI_2 warm, the circulation of the water was stopped, so that the bar was cooling slowly, at the rate of one degree in twelve minutes.

The observations were conducted in the same manner, generally, as those described in the last section. There were, however, two points of difference that should be noted.

1. The boxes were covered with *two* folds of blanket instead of *one*.
2. The bars had each *three* thermometers instead of *two*.

The outer thermometers were situated about 18 inches from the ends of the bars, the third thermometer at six inches from the centre.

The following table contains the results of the comparisons. Each pair of lines forms one comparison; the upper line in each pair being the mean of the readings of one observer, Captain Clarke, R.E.; the under, that of the second (simultaneous) observer, Quartermaster Steel, R.E. It will be understood that each number in the columns headed "Micrometer Readings" is the mean of four readings, and each number in the columns headed "Observed Temperatures," the mean of six readings.

No.	Date.	Ol_1 Micrometer Readings.	Ol_1 Obsd Temp.	Ol_1 Corrn	t_1	Ol_2 Micrometer Readings.	Ol_2 Obsd Temp.	Ol_2 Corrn	t_2
	1865.		°		°		°		°
1	Nov. 3	1251·33 h + 1258·08 k	50·43	−·34	50·09	1293·43 h + 1298·40 k	50·55	−·34	50·21
	„ „	1252·25 h + 1257·58 k	50·43	−·34	50·09	1294·05 h + 1297·65 k	50·55	−·34	50·21
2	„ „	1249·53 h + 1260·30 k	50·44	−·34	50·10	1294·53 h + 1297·30 k	50·55	−·34	50·21
	„ „	1250·68 h + 1258·75 k	50·43	−·34	50·09	1295·20 h + 1296·38 k	50·55	−·34	50·21
3	„ „	1254·40 h + 1252·90 k	50·45	−·34	50·11	1293·83 h + 1295·35 k	50·58	−·34	50·24
	„ „	1254·05 h + 1251·33 k	50·45	−·34	50·11	1294·33 h + 1293·60 k	50·59	−·34	50·25
4	„ „	1253·53 h + 1254·50 k	50·46	−·34	50·12	1295·30 h + 1294·03 k	50·59	−·34	50·25
	„ „	1252·95 h + 1253·03 k	50·45	−·34	50·11	1295·73 h + 1293·43 k	50·61	−·34	50·27
5	„ 4	1256·05 h + 1258·48 k	49·77	−·35	49·42	1292·75 h + 1299·20 k	50·01	−·32	49·69
	„ „	1255·80 h + 1256·40 k	49·77	−·35	49·42	1294·70 h + 1297·83 k	50·02	−·32	49·70
6	„ „	1256·50 h + 1256·28 k	49·78	−·35	49·43	1296·40 h + 1296·65 k	50·01	−·32	49·69
	„ „	1258·00 h + 1255·23 k	49·77	−·35	49·42	1298·28 h + 1295·53 k	50·03	−·32	49·71
7	„ „	1252·43 h + 1261·30 k	49·77	−·35	49·42	1294·73 h + 1299·98 k	50·00	−·32	49·68
	„ „	1252·08 h + 1259·93 k	49·77	−·35	49·42	1291·78 h + 1296·00 k	50·01	−·32	49·69
8	„ 6	1260·40 h + 1269·50 k	48·07	−·34	47·73	1297·00 h + 1312·50 k	48·45	−·37	48·08
	„ „	1262·85 h + 1270·43 k	48·08	−·34	47·74	1299·15 h + 1310·20 k	48·45	−·37	48·08
9	„ „	1265·03 h + 1266·18 k	48·08	−·34	47·74	1303·15 h + 1307·28 k	48·45	−·37	48·08
	„ „	1264·98 h + 1265·10 k	48·08	−·34	47·74	1304·83 h + 1305·55 k	48·45	−·37	48·08
10	„ 7	776·83 h + 777·95 k	83·72	−·18	83·54	1308·83 h + 1310·43 k	47·69	−·35	47·34
	„ „	776·90 h + 778·18 k	83·73	−·18	83·55	1308·58 h + 1308·85 k	47·69	−·35	47·34
11	„ „	774·30 h + 784·43 k	83·54	−·18	83·36	1307·15 h + 1309·05 k	47·80	−·35	47·45
	„ „	776·50 h + 783·55 k	83·54	−·18	83·36	1308·00 h + 1307·35 k	47·81	−·35	47·46
12	„ „	788·68 h + 771·98 k	83·49	−·18	83·31	1307·30 h + 1310·08 k	47·85	−·35	47·50
	„ „	790·03 h + 772·68 k	83·49	−·18	83·31	1308·30 h + 1308·90 k	47·85	−·35	47·50
13	„ 8	734·85 h + 740·00 k	86·58	−·14	86·44	1303·40 h + 1312·83 k	47·79	−·35	47·44
	„ „	734·65 h + 739·60 k	86·58	−·14	86·44	1302·43 h + 1310·70 k	47·79	−·35	47·44
14	„ „	735·90 h + 737·78 k	86·58	−·14	86·44	1306·10 h + 1307·88 k	47·85	−·35	47·50
	„ „	736·38 h + 737·28 k	86·58	−·14	86·44	1306·05 h + 1307·05 k	47·86	−·35	47·51
15	„ „	736·85 h + 737·05 k	86·55	−·14	86·41	1301·98 h + 1302·65 k	48·15	−·35	47·80
	„ „	738·33 h + 737·30 k	86·57	−·14	86·43	1302·03 h + 1300·65 k	48·15	−·35	47·80
16	„ „	738·60 h + 731·03 k	86·70	−·14	86·56	1297·80 h + 1302·55 k	48·24	−·35	47·89
	„ „	739·10 h + 729·38 k	86·71	−·14	86·57	1298·43 h + 1300·93 k	48·24	−·35	47·89
17	„ 9	730·48 h + 726·83 k	87·21	−·14	87·07	1295·88 h + 1301·08 k	48·55	−·35	48·20
	„ „	732·88 h + 726·38 k	87·22	−·14	87·08	1295·65 h + 1300·08 k	48·56	−·35	48·21
18	„ „	725·35 h + 734·88 k	87·28	−·14	87·14	1298·38 h + 1301·15 k	48·62	−·35	48·27
	„ „	723·90 h + 732·40 k	87·27	−·14	87·13	1297·85 h + 1299·63 k	48·63	−·35	48·28
19	„ „	730·78 h + 718·08 k	87·58	−·14	87·44	1297·05 h + 1296·50 k	48·89	−·35	48·54
	„ „	732·85 h + 719·03 k	87·58	−·14	87·44	1295·60 h + 1293·65 k	48·89	−·35	48·54
20	„ „	731·55 h + 722·40 k	87·52	−·14	87·38	1291·60 h + 1299·28 k	48·96	−·35	48·61
	„ „	732·15 h + 720·38 k	87·52	−·14	87·38	1292·40 h + 1297·65 k	48·96	−·35	48·61

No.	Date.	Ol_1				Ol_2			
		Micrometer Readings.	Obsd Temp.	Corrn	t_1	Micrometer Readings.	Obsd Temp.	Corrn	t_2
	1865.								
21	Nov. 10	741·73 h + 741·55 k	86°·35	−·14	86°·21	1292·50 h + 1299·13 k	48°·88	−·35	48°·53
	,, ,,	742·48 h + 740·63 k	86·34	−·14	86·20	1292·58 h + 1296·85 k	48·89	−·35	48·54
22	,, ,,	737·18 h + 740·95 k	86·56	−·14	86·42	1294·28 h + 1298·63 k	48·90	−·35	48·55
	,, ,,	736·90 h + 740·23 k	86·56	−·14	86·42	1293·18 h + 1297·95 k	48·90	−·35	48·55
23	,, ,,	737·03 h + 743·18 k	86·53	−·14	86·39	1292·25 h + 1295·18 k	49·06	−·35	48·71
	,, ,,	736·95 h + 742·08 k	86·51	−·14	86·37	1292·43 h + 1292·33 k	49·06	−·35	48·71
24	,, ,,	742·20 h + 732·58 k	86·62	−·14	86·48	1295·05 h + 1291·20 k	49·12	−·35	48·77
	,, ,,	742·05 h + 732·03 k	86·62	−·14	86·48	1293·95 h + 1289·00 k	49·12	−·35	48·77
25	,, 13	1255·45 h + 1260·48 k	47·63	−·35	47·28	796·70 h + 796·28 k	84·74	−·18	84·56
	,, ,,	1256·50 h + 1259·83 k	47·64	−·35	47·29	796·58 h + 795·15 k	84·74	−·18	84·56
26	,, ,,	1258·45 h + 1261·35 k	47·52	−·35	47 17	791·70 h + 797·45 k	84·88	−·18	84·70
	,, ,,	1258·25 h + 1260·93 k	47·53	−·35	47·18	793·63 h + 797·98 k	84·89	−·18	84·71
27	,, 14	1259·25 h + 1259·13 k	47·59	−·35	47·24	764·95 h + 777·98 k	86·46	−·14	86·32
	,, ,,	1253·08 h + 1252·43 k	47·59	−·35	47·24	764·00 h + 775·95 k	86·47	−·14	86·33
28	,, ,,	1260·23 h + 1261·55 k	47·66	−·35	47·31	774·25 h + 774·25 k	86·62	−·14	86·48
	,, ,,	1260·75 h + 1261·55 k	47·67	−·35	47·32	772·83 h + 773·65 k	86·62	−·14	86·48
29	,, ,,	1256·43 h + 1258·70 k	47·99	−·35	47·64	802·35 h + 804·38 k	84·62	−·16	84·46
	,, ,,	1256·25 h + 1258·20 k	47·99	−·35	47·64	802·05 h + 802·63 k	84·63	−·16	84·47
30	,, ,,	1255·05 h + 1259·23 k	48·06	−·35	47·71	803·25 h + 803·00 k	84·61	−·16	84·45
	,, ,,	1257·93 h + 1257·60 k	48·06	−·35	47·71	803·65 h + 801·80 k	84·62	−·16	84·46
31	,, 15	1251·18 h + 1253·85 k	48·73	−·35	48·38	772·25 h + 780·35 k	86·91	−·14	86·77
	,, ,,	1252·05 h + 1253·48 k	48·74	−·35	48·39	771·85 h + 779·15 k	86·93	−·14	86·79
32	,, ,,	1249·63 h + 1254·70 k	48·79	−·35	48·44	773·40 h + 782·38 k	86·76	−·14	86·62
	,, ,,	1249·18 h + 1254·35 k	48·79	−·35	48·44	772·98 h + 781·55 k	86·77	−·14	86·63
33	,, ,,	1246·78 h + 1250·45 k	49·14	−·35	48·79	772·13 h + 800·03 k	86·28	−·14	86·14
	,, ,,	1246·25 h + 1249·18 k	49·15	−·35	48·80	772·25 h + 799·25 k	86·29	−·14	86·15
34	,, ,,	1244·03 h + 1251·08 k	49·22	−·35	48·87	785·75 h + 789·45 k	86·22	−·14	86·08
	,, ,,	1245·53 h + 1252·28 k	49·23	−·35	48·88	785·15 h + 787·45 k	86·22	−·14	86·08
35	,, 16	1251·20 h + 1255·35 k	48·47	−·35	48·12	777·15 h + 770·98 k	86·85	−·14	86·71
	,, ,,	1252·38 h + 1255·58 k	48·47	−·35	48·12	777·15 h + 768·65 k	86·85	−·14	86·71
36	,, ,,	1252·90 h + 1253·80 k	48·59	−·35	48·24	774·25 h + 781·53 k	86·72	−·14	86·58
	,, ,,	1253·55 h + 1253·00 k	48·60	−·35	48·25	774·88 h + 779·40 k	86·74	−·14	86·60
37	,, ,,	1249·30 h + 1245·95 k	49·07	−·35	48·72	972·10 h + 987·70 k	72·14	−·18	71·96
	,, ,,	1249·38 h + 1245·83 k	49·08	−·35	48·73	970·95 h + 985·00 k	72·14	−·18	71·96
38	,, ,,	1248·50 h + 1246·48 k	49·14	−·35	48·79	1009·70 h + 1015·05 k	69·69	−·18	69·51
	,, ,,	1247·60 h + 1246·50 k	49·16	−·35	48·81	1011·65 h + 1017·10 k	69·70	−·18	69·52

Taking now the mean of the results of the two observers, that is, of each pair of lines, we get the quantities as arranged in the next Table.

No.	Difference of Length in Micrometer Divisions.	$OI_1 - OI_2$	$t_1 - 50°$	$t_2 - 50°$	$t_1 - t_2$	$21·50\,(t_1 - t_2)$	k
1	41·95 h + 40·20 k	+ 65·49	+ 0°·09	+ 0°·21	− 0°·12	− 2·58	+1·93
2	44·76 h + 37·31 k	+ 65·42	+ 0·10	+ 0·21	− 0·11	− 2·37	+2·21
3	39·86 h + 42·35 k	+ 65·54	+ 0·11	+ 0·25	− 0·14	− 3·01	+1·45
4	42·28 h + 39·96 k	+ 65·56	+ 0·12	+ 0·26	− 0·14	− 3·01	+1·43
5	37·80 h + 41·08 k	+ 62·89	− 0·58	− 0·30	− 0·28	− 6·02	+1·09
6	40·09 h + 40·33 k	+ 64·11	− 0·57	− 0·30	− 0·27	− 5·81	+0·08
7	41·00 h + 37·37 k	+ 62·47	− 0·58	− 0·31	− 0·27	− 5·81	+1·72
8	36·45 h + 41·38 k	+ 62·05	− 2·26	− 1·92	− 0·34	− 7·31	+0·64
9	38·98 h + 40·78 k	+ 63·59	− 2·26	− 1·92	− 0·34	− 7·31	−0·90
10	531·84 h + 531·57 k	+847·74	+33·55	− 2·66	+36·21	+778·52	+0·78
11	532·18 h + 524·21 k	+842·12	+33·36	− 2·54	+35·90	+771·85	−0·27
12	518·44 h + 537·16 k	+841·54	+33·31	− 2·50	+35·81	+769·92	−1·62
13	568·17 h + 571·97 k	+908·91	+36·44	− 2·56	+39·00	+838·50	−0·41
14	569·93 h + 569·93 k	+908·68	+36·44	− 2·49	+38·93	+837·00	−1·68
15	564·41 h + 564·47 k	+899·92	+36·42	− 2·20	+38·62	+830·33	+0·41
16	559·27 h + 571·53 k	+901·47	+36·57	− 2·11	+38·68	+831·62	+0·15
17	564·09 h + 573·97 k	+907·26	+37·08	− 1·79	+38·87	+835·71	−1·55
18	573·49 h + 566·75 k	+908·97	+37·14	− 1·72	+38·86	+835·49	−3·48
19	564·52 h + 576·53 k	+909·64	+37·44	− 1·46	+38·90	+836·35	−3·29
20	560·15 h + 577·08 k	+906·61	+37·38	− 1·39	+38·77	+833·56	−3·05
21	550·43 h + 556·90 k	+882·75	+36·21	− 1·46	+37·67	+809·91	−2·84
22	556·69 h + 557·70 k	+888·37	+36·42	− 1·45	+37·87	+814·21	−4·16
23	555·35 h + 551·13 k	+882·06	+36·38	− 1·29	+37·67	+809·91	−2·15
24	552·37 h + 557·79 k	+885·01	+36·48	− 1·23	+37·71	+810·77	−4·24
25	459·34 h + 464·44 k	−736·43	− 2·71	+34·56	−37·27	−801·31	+5·12
26	465·68 h + 463·42 k	−740·67	− 2·82	+34·71	−37·53	−806·90	+3·77
27	491·69 h + 478·81 k	−773·65	− 2·76	+36·33	−39·09	−840·44	+3·21
28	486·95 h + 487·60 k	−776·89	− 2·68	+36·48	−39·16	−841·94	+4·95
29	454·14 h + 454·94 k	−724·71	− 2·36	+34·47	−36·83	−791·85	+2·86
30	453·04 h + 456·02 k	−724·69	− 2·29	+34·46	−36·75	−790·13	+4·56
31	479·57 h + 473·92 k	−760·10	− 1·61	+36·78	−38·39	−825·39	+4·71
32	476·22 h + 472·56 k	−756·34	− 1·56	+36·63	−38·19	−821·09	+5·25
33	474·33 h + 450·18 k	−736·97	− 1·20	+36·15	−37·35	−803·03	+3·94
34	459·33 h + 463·23 k	−735·46	− 1·12	+36·08	−37·20	−799·80	+5·66
35	474·64 h + 485·65 k	−765·54	− 1·88	+36·71	−38·59	−829·69	+5·85
36	478·66 h + 472·93 k	−758·59	− 1·75	+36·59	−38·34	−824·31	+4·28
37	277·81 h + 259·54 k	−428·34	− 1·27	+21·96	−23·23	−499·45	−1·11
38	237·37 h + 230·41 k	−372·90	− 1·20	+19·52	−20·72	−445·48	−2·58

2.

Let the lengths of the bars OI_1 and OI_2 be expressed thus

$$OI_1 = a_1 + (t_1 - 50)\,(21 \cdot 5 + y_1) \qquad (1)$$

$$OI_2 = a_2 + (t_2 - 50)\,(21 \cdot 5 + y_2)$$

where

$21 \cdot 5 + y_1 =$ expansion of OI_1 for 1° Fahrenheit (2)

$21 \cdot 5 + y_2 =$ „ OI_2 „ „

Also let

$$70 + z = a_1 - a_2 \tag{3}$$

which is the difference of length of the bars ($\mathbf{OI}_1 - \mathbf{OI}_2$) when both are at the temperature 50° F. So that when the one bar has the temperature t_1 and the other the temperature t_2, the difference of length n is

$$n = 70 + z + (t_1 - t_2)\,21 \cdot 5 + (t_1 - 50)\,y_1 - (t_2 - 50)\,y_2$$

$$\therefore z + (t_1 - 50)\,y_1 - (t_2 - 50)\,y_2 + 70 + 21 \cdot 5\,(t_1 - t_2) - n = 0$$

Now let

$$t_1 - 50 = f_1 \tag{4}$$

$$t_2 - 50 = f_2$$

$$70 + 21 \cdot 5\,(t_1 - t_2) - n = k$$

then

$$z + f_1\,y_1 - f_2\,y_2 + k = 0 \tag{5}$$

and each comparison will supply an equation of this form.

We shall now write down the different equations, the quantities $f_1\ f_2\ k$ being given in the last table.

$$
\begin{aligned}
z &+ 0 \cdot 09\,y_1 - 0 \cdot 21\,y_2 + 1 \cdot 93 = 0 \qquad (6)\\
z &+ 0 \cdot 10\,y_1 - 0 \cdot 21\,y_2 + 2 \cdot 21 = 0\\
z &+ 0 \cdot 11\,y_1 - 0 \cdot 25\,y_2 + 1 \cdot 45 = 0\\
z &+ 0 \cdot 12\,y_1 - 0 \cdot 26\,y_2 + 1 \cdot 43 = 0\\
z &- 0 \cdot 58\,y_1 + 0 \cdot 30\,y_2 + 1 \cdot 09 = 0\\
z &- 0 \cdot 57\,y_1 + 0 \cdot 30\,y_2 + 0 \cdot 08 = 0\\
z &- 0 \cdot 58\,y_1 + 0 \cdot 31\,y_2 + 1 \cdot 72 = 0\\
z &- 2 \cdot 26\,y_1 + 1 \cdot 92\,y_2 + 0 \cdot 64 = 0\\
z &- 2 \cdot 26\,y_1 + 1 \cdot 92\,y_2 - 0 \cdot 90 = 0\\
z &+ 33 \cdot 55\,y_1 + 2 \cdot 66\,y_2 + 0 \cdot 78 = 0\\
z &+ 33 \cdot 36\,y_1 + 2 \cdot 54\,y_2 - 0 \cdot 27 = 0\\
z &+ 33 \cdot 31\,y_1 + 2 \cdot 50\,y_2 - 1 \cdot 62 = 0\\
z &+ 36 \cdot 44\,y_1 + 2 \cdot 56\,y_2 - 0 \cdot 41 = 0\\
z &+ 36 \cdot 44\,y_1 + 2 \cdot 49\,y_2 - 1 \cdot 68 = 0\\
z &+ 36 \cdot 42\,y_1 + 2 \cdot 20\,y_2 + 0 \cdot 41 = 0\\
z &+ 36 \cdot 57\,y_1 + 2 \cdot 11\,y_2 + 0 \cdot 15 = 0\\
z &+ 37 \cdot 08\,y_1 + 1 \cdot 79\,y_2 - 1 \cdot 55 = 0\\
z &+ 37 \cdot 14\,y_1 + 1 \cdot 72\,y_2 - 3 \cdot 48 = 0\\
z &+ 37 \cdot 44\,y_1 + 1 \cdot 46\,y_2 - 3 \cdot 29 = 0\\
z &+ 37 \cdot 38\,y_1 + 1 \cdot 39\,y_2 - 3 \cdot 05 = 0\\
z &+ 36 \cdot 21\,y_1 + 1 \cdot 46\,y_2 - 2 \cdot 84 = 0\\
z &+ 36 \cdot 42\,y_1 + 1 \cdot 45\,y_2 - 4 \cdot 16 = 0\\
z &+ 36 \cdot 38\,y_1 + 1 \cdot 29\,y_2 - 2 \cdot 15 = 0\\
z &+ 36 \cdot 48\,y_1 + 1 \cdot 23\,y_2 - 4 \cdot 24 = 0\\
z &- 2 \cdot 71\,y_1 - 34 \cdot 56\,y_2 + 5 \cdot 12 = 0\\
z &- 2 \cdot 82\,y_1 - 34 \cdot 71\,y_2 + 3 \cdot 77 = 0
\end{aligned}
$$

$$z - 2{\cdot}76\, y_1 - 36{\cdot}33\, y_2 + 3{\cdot}21 = 0$$
$$z - 2{\cdot}68\, y_1 - 36{\cdot}48\, y_2 + 4{\cdot}95 = 0$$
$$z - 2{\cdot}36\, y_1 - 34{\cdot}47\, y_2 + 2{\cdot}86 = 0$$
$$z - 2{\cdot}29\, y_1 - 34{\cdot}46\, y_2 + 4{\cdot}56 = 0$$
$$z - 1{\cdot}61\, y_1 - 36{\cdot}78\, y_2 + 4{\cdot}71 = 0$$
$$z - 1{\cdot}56\, y_1 - 36{\cdot}63\, y_2 + 5{\cdot}25 = 0$$
$$z - 1{\cdot}20\, y_1 - 36{\cdot}15\, y_2 + 3{\cdot}94 = 0$$
$$z - 1{\cdot}12\, y_1 - 36{\cdot}08\, y_2 + 5{\cdot}66 = 0$$
$$z - 1{\cdot}88\, y_1 - 36{\cdot}71\, y_2 + 5{\cdot}85 = 0$$
$$z - 1{\cdot}75\, y_1 - 36{\cdot}59\, y_2 + 4{\cdot}28 = 0$$
$$z - 1{\cdot}27\, y_1 - 21{\cdot}96\, y_2 - 1{\cdot}11 = 0$$
$$z - 1{\cdot}20\, y_1 - 19{\cdot}52\, y_2 - 2{\cdot}58 = 0$$

The last two of these equations we shall omit in the determination of $y_1\, y_2$ for the reason that the bar was not maintained at a constant temperature; we therefore reserve them for subsequent and special examination.

We shall see that in these comparisons the apparent or residual errors are not materially greater, when one of the bars is hot and the other cold, than in the case of both bars being cold. We therefore proceed to treat the above equations after the method of least squares.

The equations (5) multiplied in the ordinary manner give

$$\begin{aligned} 36\, z + (f_1)\, y_1 - (f_2)\, y_2 + (k) &= 0 \\ (f_1)\, z + (f_1^2)\, y_1 - (f_1 f_2)\, y_2 + (f_1 k) &= 0 \\ -(f_2)\, z - (f_1 f_2)\, y_1 + (f_2^2)\, y_2 - (f_2 k) &= 0 \end{aligned} \tag{7}$$

The necessary multiplications and additions give

$$\begin{aligned} (f_1) &= 510{\cdot}05 \\ (f_2) &= 397{\cdot}28 \\ (k) &= 36{\cdot}41 \\ (f_1^2) &= 19579{\cdot}26 \\ (f_1 f_2) &= -1906{\cdot}645 \\ (f_2^2) &= 15482{\cdot}40 \\ (f_1 k) &= -1113{\cdot}20 \\ (f_2 k) &= 1988{\cdot}13 \end{aligned} \tag{8}$$

These substituted in equations (7) give

$$\begin{aligned} 36{\cdot}00\, z + 510{\cdot}050\, y_1 - 397{\cdot}280\, y_2 + 36{\cdot}41 &= 0 \\ 510{\cdot}05\, z + 19579{\cdot}260\, y_1 + 1906{\cdot}645\, y_2 - 1113{\cdot}20 &= 0 \\ -397{\cdot}28\, z + 1906{\cdot}645\, y_1 + 15482{\cdot}400\, y_2 - 1988{\cdot}13 &= 0 \end{aligned} \tag{9}$$

If we put A, B, C, for the absolute terms, and express $z\, y_1\, y_2$ in terms of these, there results

$$\begin{aligned} z + {\cdot}1035871\, A - {\cdot}00299324\, B + {\cdot}00302667\, C &= 0 \\ y_1 - {\cdot}0029932\, A + {\cdot}00013819\, B - {\cdot}00009382\, C &= 0 \\ y_2 + {\cdot}0030267\, A - {\cdot}00009382\, B + {\cdot}00015381\, C &= 0 \end{aligned} \tag{10}$$

from which if we restore their numerical values to A, B, C, we get

$$z = -1\cdot086 : \qquad \textit{Recip. of weight} = \cdot1035871 \qquad (11)$$
$$y_1 = +0\cdot0763 : \qquad ,, \quad ,, \quad = \cdot0001382$$
$$y_2 = +0\cdot0911 : \qquad ,, \quad ,, \quad = \cdot0001538$$

These values of $z\, y_1\, y_2$ substituted in the equations of condition give the following system of errors attached to the 36 comparisons:—

Both Bars cold.		$\mathbf{OI}_1$ hot.		$\mathbf{OI}_2$ hot.	
Date.	Error.	Date.	Error.	Date.	Error.
November 3	+0·83	November 7	+2·48	November 13	+0·68
,, ,,	+1·11	,, ,,	+1·41	,, ,,	−0·69
,, ,,	+0·34	,, ,,	+0·05	,, 14	−1·39
,, ,,	+0·33	,, 8	+1·51	,, ,,	+0·33
,, 4	−0·02	,, ,,	+0·23	,, ,,	−1·55
,, ,,	−1·02	,, ,,	+2·30	,, ,,	+0·16
,, ,,	+0·62	,, ,,	+2·04	,, 15	+0·14
,, 6	−0·45	,, 9	+0·35	,, ,,	+0·71
,, ,,	−1·99	,, ,,	−1·58	,, ,,	−0·53
		,, ,,	−1·39	,, ,,	+1·20
		,, ,,	−1·17	,, 16	+1·28
		,, 10	−1·03	,, ,,	−0·28
		,, ,,	−2·34		
		,, ,,	−0·35		
		,, ,,	−2·44		

The sum of the squares of these 36 errors is 55·56; hence the probable error of a single comparison is

$$\pm \cdot674\sqrt{\frac{55\cdot56}{36-3}} = \pm\, 0\cdot875 \qquad (12)$$

and the probable errors of the determinations of $y_1\, y_2$

$$y_1 \ldots\ldots \pm\, 0\cdot875\, \sqrt{\cdot0001382} = \pm\, 0\cdot0103 \qquad (13)$$
$$y_2 \ldots\ldots \pm\, 0\cdot875\, \sqrt{\cdot0001538} = \pm\, 0\cdot0109$$

We have then finally the expansions of the two bars—

$$\textit{Expansion of } \mathbf{OI}_1 \textit{ for } 1^\circ \textit{ Fahr.} = 21\cdot5763 \pm 0\cdot0103 \qquad (14)$$
$$\textit{Expansion of } \mathbf{OI}_2 \textit{ for } 1^\circ \textit{ Fahr.} = 21\cdot5911 \pm 0\cdot0109$$

Or, if we put them in the shape of co-efficients of expansion

$$\textit{Co-efficient of Expansion of } \mathbf{OI}_1 = \cdot0000064729 \pm \cdot0000000031 \qquad (15)$$
$$\textit{Co-efficient of Expansion of } \mathbf{OI}_2 = \cdot0000064773 \pm \cdot0000000033$$

3.

The rates of expansion at which we have just arrived are—in accordance with the remark on page 220, each larger than those obtained at page 79. In the case of $\mathbf{OI}_2$ the difference is not very material, but in $\mathbf{OI}_1$ it is of some importance. It should be remarked that the bars are in all respects similar one to the other; there is no difference either in form, dimensions, material, or time and place of construction; so that there is every reason to assume *à priori* that the expansions should be equal.

Returning now to the last two of the equations of condition (6) if in them we substitute the values we have obtained for $z\, y_1\, y_2$, the residual errors are

$$-4 \cdot 30$$
$$-5 \cdot 54$$

which shows that some other value of y_2 would result from those comparisons taken by themselves. Adding the equations together we have

$$2z - 2 \cdot 47\, y_1 - 41 \cdot 48\, y_2 - 3 \cdot 69 = 0$$

substituting $z = -1 \cdot 086 : y_1 = +0 \cdot 0763$, this becomes

$$-41 \cdot 48\, y_2 - 6 \cdot 05 = 0$$
$$\therefore y_2 = -0 \cdot 146$$

giving for the expansion of $\mathbf{OI}_2$ $21 \cdot 500 - 0 \cdot 146 = 21 \cdot 354$ a result considerably in error, and in the anticipated direction.

On the evening of the 15th November, the rollers on which the bar $\mathbf{OI}_2$ was supported were jammed so as to prevent their revolution. The object of this was to ascertain whether the friction so produced would on the heating of the bar the following morning diminish the expansion, as the bar under these circumstances would expand by sliding over the fixed rollers. The errors exhibited in the two first comparisons on the 16th show that this produced no effect whatever.

4.

An examination of the simultaneous micrometer readings of the two observers as recorded in the expansion experiments in this and the preceding section brings to light a certain amount of personal error. In each comparison the mean readings of each bar are given in the form

$$a h + a' k$$
$$b h + b' k$$

where a, a', b, b', are each the mean of four micrometer readings; the first two appertaining to the observer A, the second two to the observer B. It will be remembered that $a\, h$ and $b'\, k$ are simultaneous readings, as also are $b\, h$ and $a'\, k$. If z be the distance of the zeros of the microscopes, and u the length of the bar, then, but for errors of observation,

$$u + a h + b' k - z = 0$$
$$u + b h + a' k - z = 0$$

If the bar remained absolutely undisturbed between the two pairs of observations these equations represent, that is to say, while the observers change places, then would also

$$u + a\,h + a'\,k - z = 0$$
$$u + b\,h + b'\,k - z = 0$$

hold good. But these do *not* always, as the bar was sometimes readjusted in focus or otherwise, besides being possibly shaken in the readings of the thermometers which took place in the interval. In the ordinary comparisons where there is only one observer, the thermometers are not read, nor the box containing the bars touched in any way, between the reading of the micrometers.

Now let the personal error of the observer A in reading the line under the micrometer H be e_a; and the personal error in reading the line under K, e_α, while e_b, e_β represent the corresponding personal errors of B. Then we have

$$e_b + e_\alpha - e_a - e_\beta = (b - a)\,h + (a' - b')\,k = E$$

We now abstract this quantity E for each comparison, and place them in the following Tables :—

BRONZE BAR. 1st Series.

Date.	E.	Date.	E.	Date.	E.	Date.	E.
1865.		1865.		1865.		1865.	
Feb. 17	+1·27	Feb. 25	+1·62	March 4	+2·57	March 9	+3·01
„ 18	1·31	„ „	2·53	„ 6	3·92	„ „	3·33
„ „	2·71	„ 27	4·61	„ „	2·45	„ „	2·77
„ 20	1·59	„ „	1·85	„ „	3·03	„ 10	2·03
„ „	2·13	„ 28	3·19	„ „	1·64	„ „	2·93
„ 21	1·13	„ „	2·04	„ „	2·07	„ „	2·27
„ „	1·75	„ „	1·19	„ 7	4·13	„ 11	2·82
„ „	2·73	March 3	1·77	„ „	2·50	„ „	2·13
„ 22	1·08	„ „	3·08	„ „	4·37	„ „	2·79
„ 24	2·13	„ 4	2·50	„ „	3·93	„ „	1·79
„ „	2·96	„ „	3·12	„ 8	2·41	„ „	+1·55
„ 25	1·45	„ „	2·43	„ „	3·21		
„ „	+1·23	„ „	+2·65	„ „	+2·91		

BRONZE BAR. 2nd Series.

Date.	E.	Date.	E.	Date.	E.	Date.	E.
1865.		1865.		1865.		1865.	
April 29	+1·78	May 3	+2·34	May 5	+2·39	May 9	+2·61
„ „	0·77	„ „	1·59	„ „	2·98	„ „	1·73
„ „	1·87	„ „	1·98	„ 6	2·03	„ „	1·94
„ „	1·08	„ 4	2·07	„ „	2·97	„ „	3·39
May 1	2·11	„ „	2·42	„ „	1·42	„ 10	2·94
„ „	1·98	„ 5	2·53	„ „	2·57	„ „	+1·61
„ 3	+2·35	„ „	+2·95	„ 8	+3·55		

Steel Bar. 1st Series.

Date.	E.	Date.	E.	Date.	E.	Date.	E.
1865.		1865.		1865.		1865.	
Feb. 17	+0·16	Feb. 25	+2·42	March 4	+2·57	March 9	+3·43
„ 18	0·60	„ „	1·87	„ 6	3·66	„ „	1·83
„ „	1·39	„ 27	1·70	„ „	2·39	„ „	2·22
„ 20	3·35	„ „	2·67	„ „	4·01	„ 10	3·92
„ „	2·89	„ 28	1·67	„ „	2·73	„ „	3·03
„ 21	2·26	„ „	0·74	„ „	1·75	„ „	2·78
„ „	0·85	„ „	1·47	„ 7	2·63	„ 11	2·91
„ „	2·81	March 3	1·74	„ „	1·28	„ „	0·83
„ 22	3·15	„ „	2·59	„ „	2·64	„ „	1·67
„ 24	2·09	„ 4	1·74	„ „	2·81	„ „	2·38
„ „	1·12	„ „	1·82	„ 8	2·22	„ „	+1·37
„ 25	2·09	„ „	1·80	„ „	3·88		
„ „	+0·92	„ „	+1·65	„ „	+2·93		

Steel Bar. 2nd Series.

Date.	E.	Date.	E.	Date.	E.	Date.	E.
1865.		1865.		1865.		1865.	
April 29	+0·60	May 3	+1·71	May 5	+2·43	May 9	+2·66
„ „	1·09	„ „	1·50	„ „	2·06	„ „	2·67
„ „	1·39	„ „	2·93	„ 6	2·37	„ „	2·55
„ „	0·68	„ 4	1·51	„ „	1·97	„ „	3·27
May 1	0·90	„ „	1·22	„ „	1·61	„ 10	1·04
„ „	1·95	„ 5	2·61	„ „	2·29	„ „	+2·53
„ 3	+1·98	„ „	+2·55	„ 8	+3·33		

Ol_1

Date.	E.	Date.	E.	Date.	E.	Date.	E.
1865.		1865.		1865.		1865.	
Nov. 3	+1·13	Nov. 7	+2·46	Nov. 10	+1·33	Nov. 15	+0·99
„ „	+2·15	„ „	0·52	„ „	0·35	„ ,	−0·08
„ „	+0·97	„ 8	0·16	„ „	0·81	„ „	+0·59
„ „	+0·71	„ „	0·78	„ „	0·32	„ „	+0·24
„ 4	+1·46	„ „	0·98	„ 13	1·36	„ 16	+0·76
„ „	+2·03	„ „	1·71	„ „	0·18	„ „	+1·16
„ „	+0·81	„ 9	2·27	„ 14	0·42	„ „	+0·16
„ 6	+1·21	„ „	0·82	„ „	0·41	„ „	−0·73
„ „	+0·82	„ „	0·89	„ „	0·26		
„ 7	−0·13	„ „	+2·09	„ „	+3·59		

OI_2

Date.		E.	Date.		E.	Date.		E.	Date.		E.
1865.			1865.			1865.			1865.		
Nov.	3	+1·09	Nov.	7	+2·03	Nov.	10	+1·88	Nov.	15	+0·64
"	"	1·27	"	"	1·74	"	"	−0·33	"	"	+0·33
"	"	1·80	"	8	0·92	"	"	+2·42	"	"	+0·72
"	"	0·82	"	"	0·62	"	"	+0·88	"	"	+1·12
"	4	2·65	"	"	1·63	"	13	+0·80	"	16	+1·86
"	"	2·39	"	"	1·79	"	"	+1·12	"	"	+2·20
"	"	0·82	"	9	0·61	"	14	+0·86	"	"	+1·23
"	6	3·55	"	"	0·79	"	"	−0·65	"	"	−0·08
"	"	2·72	"	"	1·12	"	"	+1·16			
"	7	+1·06	"	"	+1·94	"	"	+1·27			

From these tables it appears that the value of E is positive in every instance in the case of the Indian Bars. In the case of either of the Iron Bars there are three negative values against 35 positive values. In the second series of experiments on either of the Indian Bars the values are rather smaller than in the first series. The mean results are—

Bronze . . . . $e_a - e_\alpha + e_b - e_\beta = 2\cdot 34$

Steel . . . . $e_a - e_\alpha + e_b - e_\beta = 2\cdot 10$

OI_1 . . . . $e_a - e_\alpha + e_b - e_\beta = 0\cdot 95$

OI_2 . . . . $e_a - e_\alpha + e_b - e_\beta = 1\cdot 28$

Here two facts present themselves at once,—(1) the amount of personal error exhibited in the case of the Indian Bars is twice as great as in the others; (2) the error is the same in sign for all four bars. These might be accounted for on the hypothesis that the "bisection" of the observer A was always to the right of the bisection by the observer B—while the actual amount of difference varies with different lines. If we further suppose that each of the two lines on a bar produce the same amount of personal error, then since the micrometers measure in opposite directions, $e_a = -e_\alpha$, $e_\beta = -e_b$, and in the case of the four bars we have the results—

$e_b - e_a = 1\cdot 17$ $\qquad$ $e_b - e_a = 0\cdot 48$

$e_b - e_a = 1\cdot 05$ $\qquad$ $e_b - e_a = 0\cdot 64$

If we further suppose the two observers to have equal personal errors in opposite directions, then in the four different bars—

$e_a = -e_b = -0\cdot 58$ $\qquad$ $e_a = -e_b = -0\cdot 24$

$e_a = -e_b = -0\cdot 52$ $\qquad$ $e_a = -e_b = -0\cdot 32$

And indeed if we examine the observations as recorded, it will be seen that in almost every instance the H readings of the first observer are less, and his K readings greater, than those of the second observer. Where this does not hold it is legitimate to suppose that the bar has been shaken or adjusted between the two observations.

In order to examine further into the matter, some special observations were made on OI_1 and OI_2 with the microscopes interchanged. The readings were made in the same manner as in the expansion experiments. Each number in the following tables is the mean of four readings, and the six lines in each group correspond to one visit to the bars.

OI_1

Date.		Captain Clarke, R.E.	Quartermaster Steel.	E.	Remarks.
1865.					
Nov.	28	4·05 *k* + 6·22 *h*	4·12 *k* + 5·40 *h*	+0·71	K left, H right.
,,	,,	3·82 *k* + 6·60 *h*	4·17 *k* + 6·27 *h*	+0·54	
,,	,,	3·72 *k* + 6·50 *h*	2·90 *k* + 6·42 *h*	−0·59	
,,	,,	3·00 *k* + 7·10 *h*	2·45 *k* + 7·15 *h*	−0·48	
,,	,,	2·75 *k* + 7·75 *h*	2·92 *k* + 7·65 *h*	+0·21	
,,	,,	2·45 *k* + 7·65 *h*	1·62 *k* + 7·55 *h*	−0·58	
,,	,,	6·45 *k* + 1·60 *h*	6·15 *k* + 1·22 *h*	+0·06	,, ,,
,,	,,	6·62 *k* + 1·68 *h*	6·37 *k* + 1·00 *h*	+0·34	
,,	,,	6·08 *k* + 1·35 *h*	5·77 *k* + 1·40 *h*	−0·29	
,,	,,	5·52 *k* + 1·80 *h*	5·15 *k* + 1·48 *h*	−0·04	
,,	,,	5·70 *k* + 1·72 *h*	5·02 *k* + 1·78 *h*	−0·59	
,,	,,	5·65 *k* + 1·88 *h*	5·55 *k* + 1·40 *h*	+0·30	
,,	29	9·32 *h* + 8·18 *k*	8·85 *h* + 8·00 *k*	−0·23	H left, K right.
,,	,,	9·40 *h* + 7·92 *k*	8·77 *h* + 8·05 *k*	−0·60	
,,	,,	9·22 *h* + 7·88 *k*	8·52 *h* + 7·57 *k*	−0·31	
,,	,,	8·82 *h* + 8·02 *k*	8·05 *h* + 7·72 *k*	−0·37	
,,	,,	9·00 *h* + 8·32 *k*	8·50 *h* + 7·97 *k*	−0·12	
,,	,,	8·62 *h* + 8·28 *k*	8·25 *h* + 8·17 *k*	−0·21	
,,	30	2·02 *h* + 4·32 *k*	2·75 *h* + 3·07 *k*	+1·58	,, ,,
,,	,,	2·05 *h* + 4·55 *k*	1·62 *h* + 3·50 *k*	+0·49	
,,	,,	2·02 *h* + 4·80 *k*	1·77 *h* + 3·95 *k*	+0·48	
,,	,,	2·05 *h* + 5·62 *k*	1·30 *h* + 4·72 *k*	+0·12	
,,	,,	2·08 *h* + 5·50 *k*	1·55 *h* + 4·40 *k*	+0·45	
,,	,,	1·78 *h* + 5·75 *k*	1·47 *h* + 4·42 *k*	+0·81	

OI_2

Nov.	30	6·08 *h* + 9·18 *k*	6·80 *h* + 7·70 *k*	+1·75	H left, K right.
,,	,,	5·62 *h* + 10·02 *k*	6·02 *h* + 7·80 *k*	+2·09	
,,	,,	5·68 *h* + 9·30 *k*	5·90 *h* + 8·30 *k*	+0·97	
,,	,,	5·10 *h* + 9·75 *k*	5·85 *h* + 8·40 *k*	+1·67	
,,	,,	5·55 *h* + 9·75 *k*	6·02 *h* + 8·37 *k*	+1·47	
,,	,,	5·02 *h* + 9·52 *k*	5·30 *h* + 8·50 *k*	+1·04	
Dec.	1	4·40 *h* + 5·60 *k*	3·70 *h* + 4·85 *k*	+0·04	,, ,,
,,	,,	4·40 *h* + 5·85 *k*	3·67 *h* + 5·00 *k*	+0·10	
,,	,,	4·28 *h* + 5·92 *k*	4·72 *h* + 4·87 *k*	+1·19	
,,	,,	3·48 *h* + 6·10 *k*	3·75 *h* + 5·27 *k*	+0·87	
,,	,,	3·75 *h* + 6·25 *k*	4·07 *h* + 5·60 *k*	+0·77	
,,	,,	3·60 *h* + 6·10 *k*	3·30 *h* + 5·07 *k*	+0·58	
,,	,,	0·78 *k* + 6·80 *h*	1·80 *k* + 5·00 *h*	+2·25	K left, H right.
,,	,,	0·75 *k* + 6·52 *h*	1·52 *k* + 5·25 *h*	+1·62	
,,	,,	0·50 *k* + 6·68 *h*	0·60 *k* + 5·17 *h*	+1·28	
,,	,,	0·98 *k* + 6·90 *h*	0·32 *k* + 4·82 *h*	+1·13	
,,	,,	0·78 *k* + 6·58 *h*	0·42 *k* + 4·87 *h*	+1·08	
,,	,,	0·68 *k* + 6·60 *h*	0·95 *k* + 5·05 *h*	+1·45	
,,	2	7·58 *k* + 8·12 *h*	7·78 *k* + 6·45 *h*	+1·49	,, ,,
,,	,,	7·55 *k* + 8·00 *h*	8·25 *k* + 7·07 *h*	+1·30	
,,	,,	7·83 *k* + 8·20 *h*	7·80 *k* + 6·75 *h*	+1·13	
,,	,,	7·55 *k* + 8·12 *h*	8·62 *k* + 6·57 *h*	+2·09	
,,	,,	7·52 *k* + 8·30 *h*	6·75 *k* + 6·78 *h*	+0·60	
,,	,,	8·10 *k* + 8·42 *h*	7·72 *k* + 6·67 *h*	+1·09	

In the first of these tables, column headed E, there are as many positive as negative signs, and the average of the values of E is + 0·07. The result is not materially different whether H be to the left and K to the right, or *vice versâ*. This value is sensibly different from that obtained from the expansion experiments.

In the second Table containing the readings of $\mathbf{Ol}_2$ it appears that E is in every case positive. The mean of the first 12 values, when H is on the left and K on the right, is + 1·05; and of the second 12, when H is on the right and K on the left, the mean is + 1·38. In these experiments the bar may safely be assumed to have remained absolutely unmoved during the readings of each visit (or group of six lines), so that we can get the difference of readings of the two observers for *each* line; taking the means in 12 lines each, we get—

Captain Clarke.	Quartermaster Steel.	Position of Micrometer.
4·75 *h* + 7·78 *k*	4·93 *h* + 6·64 *k*	H on the left, K right.
4·22 *k* + 7·44 *h*	4·38 *k* + 5·87 *h*	K on the left, H right.

Where each number is the mean of 48 readings. Here the difference of the personal equations of the two observers is—

Left Line.	Right Line.	Position of Micrometer.
+ 0·18 *h*	− 1·14 *k*	H on the left, K right,
+ 0·16 *k*	− 1·57 *h*	K on the left, H right.

Thus it appears that the discrepancy arises almost entirely from the manner of bisecting one—the right—line. The mean value of E from all these observations on $\mathbf{Ol}_2$ = + 1·21 which agrees with the value derived from the expansion experiments.

But in the case of $\mathbf{Ol}_1$ we have found a different value. This leads us to conclude that a line may be read or bisected differently at one time from another. That is to say, the line may have a different appearance to one or both observers at different times. This may be the case actually, owing to very minute particles of dust hanging about the edges of the line, which are not easily dislodged. It is impossible to protect at all times the polished surfaces from dust, nor ought they to be cleaned oftener than is absolutely necessary.

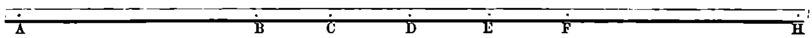
The lines on the Indian Bars, which bring out the greatest amount of personal error, are fine and some of them rather faint; the lines on $\mathbf{Ol}_1$ $\mathbf{Ol}_2$ are dark and strong, but they have straight parallel edges. The parallelism of their edges is certainly of more importance that the fineness of the lines. In the platinum metre, for instance, the lines are excessively fine, but they have no sharp parallel edges, and consequently are most unfavourable for observation.

XVIII.

DETERMINATION OF THE LENGTH OF THE INDIAN STANDARD

I_S

The Indian Standard Bar I_S has seven points on its upper surface distributed as shown in the diagram of OI_1, page 81. They are marked with the distinctive letters A B C D E F H, thus



The comparisons made were the following :—

[A · B]	with	Y_{55}	:	[A · B]	with	[B · E]	:	[B · D]	with	[D · F]
[B · E]	,,	Y_{55}		[B · E]	,,	[C · F]		[D · F]	,,	[C · E]
[C · F]	,,	Y_{55}		[C · F]	,,	[F · H]		[B · C]	,,	[C · D]
[F · H]	,,	Y_{55}		[B · D]	,,	[C · E]		[D · E]	,,	[E · F]

The observations are recorded in the usual form in the following Tables :—

COMPARISONS OF THE DIFFERENT SPACES ON THE STEEL BAR WITH ONE ANOTHER, AND WITH THE STANDARD YARD Y_{55}.

Date.	[A · B]	[B · E]	Difference of Length in Micrometer Divisions.	[A·B]−[B·E]
1865.				
May 25	16·93 *h* + 27·58 *k*	44·28 *h* + 44·63 *k*	27·35 *h* + 17·05 *k*	35·38
,, ,,	22·28 *h* + 21·57 *k*	43·23 *h* + 43·63 *k*	20·95 *h* + 22·06 *k*	34·29
,, ,,	18·43 *h* + 25·30 *k*	39·78 *h* + 47·25 *k*	21·35 *h* + 21·95 *k*	34·52
,, ,,	19·63 *h* + 22·91 *k*	29·10 *h* + 57·25 *k*	9·47 *h* + 34·34 *k*	34·96
,, ,,	23·65 *h* + 18·33 *k*	45·00 *h* + 40·28 *k*	21·35 *h* + 21·95 *k*	34·52
,, ,,	21·68 *h* + 19·73 *k*	42·03 *h* + 42·98 *k*	20·35 *h* + 23·25 *k*	34·76
,, ,,	23·65 *h* + 16·23 *k*	45·22 *h* + 37·07 *k*	21·57 *h* + 20·84 *k*	33·81
,, ,,	24·00 *h* + 15·87 *k*	37·52 *h* + 46·22 *k*	13·52 *h* + 30·35 *k*	35·00
,, 26	21·63 *h* + 26·60 *k*	49·57 *h* + 41·55 *k*	27·94 *h* + 14·95 *k*	34·17
,, ,,	24·88 *h* + 24·03 *k*	54·00 *h* + 37·80 *k*	29·12 *h* + 13·77 *k*	34·17

Date.	[B·E]	[C·F]	Difference of Length in Micrometer Divisions.	[C·F]−[B·E]
1865. May 20	32·43 *h* + 30·08 *k*	35·18 *h* + 22·70 *k*	− 2·75 *h* + 7·38 *k*	3·71
„ 22	23·10 *h* + 31·58 *k*	24·50 *h* + 24·90 *k*	− 1·40 *h* + 6·68 *k*	4·22
„ „	29·42 *h* + 26·03 *k*	29·20 *h* + 21·68 *k*	+ 0·22 *h* + 4·35 *k*	3·65
„ „	26·55 *h* + 28·57 *k*	24·18 *h* + 26·48 *k*	+ 2·37 *h* + 2·09 *k*	3·55
„ „	25·84 *h* + 28·85 *k*	28·07 *h* + 21·67 *k*	− 2·23 *h* + 7·18 *k*	3·96
„ „	25·85 *h* + 28·70 *k*	22·93 *h* + 25·95 *k*	+ 2·92 *h* + 2·75 *k*	4·52
„ „	28·08 *h* + 23·77 *k*	23·20 *h* + 23·28 *k*	+ 4·88 *h* + 0·49 *k*	4·27
„ „	26·73 *h* + 24·57 *k*	26·18 *h* + 20·32 *k*	+ 0·55 *h* + 4·25 *k*	3·83
„ „	20·73 *h* + 25·95 *k*	22·95 *h* + 21·53 *k*	− 2·22 *h* + 4·42 *k*	1·76
„ „	20·47 *h* + 27·57 *k*	20·32 *h* + 24·73 *k*	+ 0·15 *h* + 2·84 *k*	2·39
„ 23	29·72 *h* + 20·76 *k*	29·90 *h* + 14·82 *k*	− 0·18 *h* + 5·94 *k*	4·60
„ „	25·21 *h* + 25·56 *k*	23·76 *h* + 21·86 *k*	+ 1·45 *h* + 3·70 *k*	4·11
„ „	29·07 *h* + 27·32 *k*	28·83 *h* + 24·72 *k*	+ 0·24 *h* + 2·60 *k*	2·27
„ „	29·88 *h* + 26·85 *k*	27·10 *h* + 26·20 *k*	+ 2·78 *h* + 0·65 *k*	2·73
„ „	28·90 *h* + 27·91 *k*	26·45 *h* + 26·38 *k*	+ 2·45 *h* + 1·53 *k*	3·17
„ „	24·63 *h* + 29·10 *k*	23·85 *h* + 27·23 *k*	+ 0·78 *h* + 1·87 *k*	2·11
„ „	24·93 *h* + 27·55 *k*	23·87 *h* + 25·89 *k*	+ 1·06 *h* + 1·66 *k*	2·17
„ „	26·78 *h* + 25·40 *k*	23·82 *h* + 24·45 *k*	+ 2·96 *h* + 0·95 *k*	3·11
„ „	26·98 *h* + 23·23 *k*	25·25 *h* + 20·30 *k*	+ 1·73 *h* + 2·93 *k*	3·72
„ „	27·70 *h* + 23·03 *k*	24·63 *h* + 22·18 *k*	+ 3·07 *h* + 0·85 *k*	3·12

Date.	[C·F]	[F·H]	Difference of Length in Micrometer Divisions.	[F·H]−[C·F]
1865.				
May 26	60·83 *h* + 59·33 *k*	29·23 *h* + 18·63 *k*	31·60 *h* + 40·70 *k*	57·65
„ „	62·78 *h* + 58·10 *k*	24·52 *h* + 24·40 *k*	38·26 *h* + 33·70 *k*	57·36
„ „	60·55 *h* + 59·18 *k*	24·42 *h* + 24·47 *k*	36·13 *h* + 34·71 *k*	56·47
„ „	60·87 *h* + 60·13 *k*	23·63 *h* + 23·85 *k*	37·24 *h* + 36·28 *k*	58·61
„ 27	59·27 *h* + 62·98 *k*	24·80 *h* + 25·51 *k*	34·47 *h* + 37·47 *k*	57·35
„ „	58·60 *h* + 63·47 *k*	24·48 *h* + 25·12 *k*	34·12 *h* + 38·35 *k*	57·78
„ „	62·01 *h* + 58·15 *k*	23·51 *h* + 24·90 *k*	38·50 *h* + 33·25 *k*	57·19
„ „	58·11 *h* + 61·18 *k*	25·43 *h* + 23·45 *k*	32·68 *h* + 37·73 *k*	56·14
„ „	62·43 *h* + 55·93 *k*	28·15 *h* + 19·95 *k*	34·28 *h* + 35·98 *k*	56·01
„ „	66·25 *h* + 52·05 *k*	28·48 *h* + 19·12 *k*	37·77 *h* + 32·93 *k*	56·39

Date.	[B·D]	[C·E]	Difference of Length in Micrometer Divisions.	[C·E]+[B·D]
1865.				
July 1	77·40 *h* + 81·33 *k*	56·63 *h* + 59·52 *k*	20·77 *h* + 21·81 *k*	33·95
„ „	80·83 *h* + 76·57 *k*	55·95 *h* + 61·11 *k*	24·88 *h* + 15·46 *k*	32·14
„ „	79·47 *h* + 80·83 *k*	64·03 *h* + 55·10 *k*	15·44 *h* + 25·73 *k*	32·84
„ „	79·85 *h* + 77·98 *k*	57·97 *h* + 60·10 *k*	21·88 *h* + 17·88 *k*	31·69
„ „	79·70 *h* + 78·47 *k*	54·10 *h* + 62·83 *k*	25·60 *h* + 15·64 *k*	32·86
„ „	79·62 *h* + 82·98 *k*	67·32 *h* + 53·68 *k*	12·30 *h* + 29·30 *k*	33·19
„ 3	78·70 *h* + 86·25 *k*	63·37 *h* + 61·20 *k*	15·33 *h* + 25·05 *k*	32·20
„ „	80·70 *h* + 82·37 *k*	64·32 *h* + 58·12 *k*	16·38 *h* + 24·25 *k*	32·40
„ „	84·30 *h* + 78·52 *k*	61·15 *h* + 61·38 *k*	23·15 *h* + 17·14 *k*	32·11
„ „	77·03 *h* + 83·92 *k*	62·23 *h* + 59·35 *k*	14·80 *h* + 24·57 *k*	31·40

Date.		[B·D]	[D·F]	Difference of Length in Micrometer Divisions.	[D·F] — [B·D]
1865.					
May	27	48·78 *h* + 41·38 *k*	22·60 *h* + 26·67 *k*	26·18 *h* + 14·71 *k*	32·58
,,	,,	46·65 *h* + 42·97 *k*	28·27 *h* + 21·60 *k*	18·38 *h* + 21·37 *k*	31·69
,,	,,	45·53 *h* + 43·48 *k*	17·30 *h* + 31·03 *k*	28·23 *h* + 12·45 *k*	32·41
,,	,,	49·27 *h* + 40·61 *k*	21·27 *h* + 27·35 *k*	28·00 *h* + 13·26 *k*	32·87
,,	,,	45·70 *h* + 42·05 *k*	22·40 *h* + 25·30 *k*	23·30 *h* + 16·75 *k*	31·92
,,	,,	40·55 *h* + 46·57 *k*	23·15 *h* + 24·88 *k*	17·40 *h* + 21·69 *k*	31·17
,,	29	43·82 *h* + 45·80 *k*	25·30 *h* + 23·85 *k*	18·52 *h* + 21·95 *k*	32·27
,,	,,	45·53 *h* + 44·58 *k*	28·98 *h* + 20·47 *k*	16·55 *h* + 24·11 *k*	32·43
,,	,,	47·42 *h* + 42·20 *k*	21·80 *h* + 27·28 *k*	25·62 *h* + 14·92 *k*	32·30
,,	,,	46·78 *h* + 42·52 *k*	25·23 *h* + 24·17 *k*	21·49 *h* + 18·35 *k*	31·76

Date.		[D·F]	[C·E]	Difference of Length in Micrometer Divisions.	[C·E] — [D·F]
1865.					
July	1	62·68 *h* + 55·58 *k*	64·10 *h* + 51·95 *k*	−1·42 *h* + 3·63 *k*	1·77
,,	,,	61·27 *h* + 57·08 *k*	60·15 *h* + 57·13 *k*	1·12 *h* − 0·05 *k*	0·85
,,	,,	63·78 *h* + 55·05 *k*	58·85 *h* + 59·22 *k*	4·93 *h* − 4·17 *k*	0·59
,,	,,	59·45 *h* + 58·78 *k*	60·15 *h* + 57·73 *k*	−0·70 *h* + 1·05 *k*	0·28
,,	,,	62·57 *h* + 55·02 *k*	60·85 *h* + 56·13 *k*	1·72 *h* − 1·11 *k*	0·48
,,	,,	68·92 *h* + 53·17 *k*	72·35 *h* + 48·80 *k*	−3·43 *h* + 4·37 *k*	0·76
,,	3	64·00 *h* + 60·67 *k*	65·47 *h* + 58·00 *k*	−1·47 *h* + 2·67 *k*	0·96
,,	,,	62·13 *h* + 61·58 *k*	65·57 *h* + 55·55 *k*	−3·44 *h* + 6·03 *k*	2·08
,,	,,	69·98 *h* + 54·33 *k*	71·48 *h* + 52·07 *k*	−1·50 *h* + 2·26 *k*	0·61
,,	,,	58·88 *h* + 63·18 *k*	59·78 *h* + 61·47 *k*	−0·90 *h* + 1·71 *k*	0·65

Date.		[B·C]	[C·D]	Difference of Length in Micrometer Divisions.	[C·D] — [B·C]
1865.					
May	29	47·78 *h* + 44·75 *k*	45·08 *h* + 41·92 *k*	2·70 *h* + 2·83 *k*	4·41
,,	,,	45·71 *h* + 47·20 *k*	45·08 *h* + 41·38 *k*	0·63 *h* + 5·82 *k*	5·15
,,	,,	45·15 *h* + 44·28 *k*	41·43 *h* + 44·41 *k*	3·72 *h* − 0·13 *k*	2·86
,,	,,	44·28 *h* + 45·88 *k*	43·10 *h* + 43·65 *k*	1·18 *h* + 2·23 *k*	2·72
,,	,,	42·62 *h* + 49·95 *k*	39·05 *h* + 47·78 *k*	3·57 *h* + 2·17 *k*	4·57
,,	,,	43·30 *h* + 49·32 *k*	41·13 *h* + 47·06 *k*	2·17 *h* + 2·26 *k*	3·53
,,	,,	43·98 *h* + 48·25 *k*	45·53 *h* + 41·52 *k*	−1·55 *h* + 6·73 *k*	4·14
,,	,,	45·53 *h* + 46·22 *k*	43·72 *h* + 42·78 *k*	1·81 *h* + 3·44 *k*	4·19
,,	,,	46·55 *h* + 44·57 *k*	47·72 *h* + 38·12 *k*	−1·17 *h* + 6·45 *k*	4·22
,,	,,	46·52 *h* + 44·65 *k*	46·28 *h* + 39·50 *k*	0·24 *h* + 5·15 *k*	4·30

Date.	[D · E]	[E · F]	Difference of Length in Micrometer Divisions.	[D · E] — [E · F]
1865.				
May 30	26·88 h + 21·67 k	44·47 h + 43·63 k	17·59 h + 21·96 k	31·54
,, ,,	24·25 h + 21·85 k	45·07 h + 41·63 k	20·82 h + 19·78 k	32·36
,, ,,	23·33 h + 24·55 k	37·93 h + 50·42 k	14·60 h + 25·87 k	32·28
,, ,,	21·22 h + 26·72 k	44·77 h + 42·82 k	23·55 h + 16·10 k	31·60
,, ,,	21·75 h + 26·80 k	43·51 h + 43·42 k	21·76 h + 16·62 k	30·59
,, ,,	24·65 h + 23·40 k	40·40 h + 45·23 k	15·75 h + 21·83 k	29·97
,, ,,	24·42 h + 23·95 k	35·43 h + 50·53 k	11·01 h + 26·58 k	29·99
,, ,,	23·72 h + 26·00 k	43·48 h + 43·73 k	19·76 h + 17·73 k	29·88
,, ,,	25·83 h + 22·50 k	43·45 h + 41·80 k	17·62 h + 19·30 k	29·43
,, ,,	25·93 h + 22·30 k	38·68 h + 47·07 k	12·75 h + 24·77 k	29·93

Date.	Temp.	[A · B]	Y_{55}	Difference of Length in Micrometer Divisions.	[A · B] — Y_{55}
1865.	°				
June 22	62·48	82·52 h + 81·65 k	97·63 h + 102·77 k	15·11 h + 21·12 k	28·89
,, ,,	62·60	73·33 h + 89·85 k	88·32 h + 111·63 k	14·99 h + 21·78 k	29·32
,, ,,	62·74	79·70 h + 82·23 k	97·42 h + 101·45 k	17·72 h + 19·22 k	29·45
,, ,,	62·82	78·43 h + 83·58 k	100·53 h + 96·87 k	22·10 h + 13·29 k	28·20
,, 23	62·83	86·80 h + 76·32 k	100·03 h + 98·93 k	13·23 h + 22·61 k	28·58
,, ,,	62·87	85·82 h + 76·30 k	96·83 h + 101·08 k	11·01 h + 24·78 k	28·55
,, ,,	63·06	74·88 h + 86·57 k	92·75 h + 105·32 k	17·87 h + 18·75 k	29·19
,, ,,	63·15	83·45 h + 77·57 k	96·30 h + 101·07 k	12·85 h + 23·50 k	28·99
,, ,,	63·27	77·47 h + 81·77 k	98·27 h + 97·72 k	20·80 h + 15·95 k	29·29
,, ,,	63·33	77·78 h + 81·05 k	98·57 h + 96·95 k	20·79 h + 15·90 k	29·24

Date.	Temp.	[B · E]	Y_{55}	Difference of Length in Micrometer Divisions.	Y_{55} — [B · E]
1865.	°				
April 8	47·33	25·68 h + 28·83 k	24·88 h + 28·28 k	0·80 h + 0·55 k	1·08
,, ,,	47·27	17·00 h + 36·45 k	16·80 h + 39·12 k	0·20 h − 2·67 k	−1·97
,, ,,	47·33	23·65 h + 28·72 k	22·48 h + 29·62 k	1·17 h − 0·90 k	0·21
,, ,,	47·35	27·57 h + 25·28 k	30·82 h + 21·52 k	− 3·25 h + 3·76 k	0·42
,, 10	47·85	23·42 h + 27·45 k	24·85 h + 25·68 k	− 1·43 h + 1·77 k	0·28
,, ,,	48·03	24·25 h + 19·98 k	26·00 h + 17·22 k	− 1·75 h + 2·76 k	0·81
,, ,,	48·25	22·48 h + 20·65 k	22·32 h + 20·07 k	0·16 h + 0·58 k	0·59
,, ,,	48.40	26·18 h + 13·32 k	23·92 h + 14·68 k	2·26 h − 1·36 k	0·71
,, 11	48·88	21·03 h + 16·90 k	20·65 h + 16·83 k	0·38 h + 0·07 k	0·36
,, ,,	49·09	15·12 h + 22·87 k	15·57 h + 21·38 k	− 0·45 h + 1·49 k	0·83
June 1	61·04	29·83 h + 35·70 k	28·22 h + 31·37 k	1·61 h + 4·33 k	4·74
,, ,,	61·17	29·00 h + 36·50 k	28·22 h + 30·72 k	0·78 h + 5·78 k	5·24
,, ,,	61·23	32·30 h + 37·38 k	34·02 h + 29·55 k	− 1·72 h + 7·83 k	4·88
,, 2	60·83	39·83 h + 30·80 k	30·70 h + 34·02 k	9·13 h − 3·22 k	4·69
,, ,,	60·79	35·77 h + 35·23 k	35·53 h + 29·22 k	0·24 h + 6·01 k	4·99
,, ,,	60·76	36·43 h + 37·40 k	37·38 h + 29·27 k	− 0·95 h + 8·13 k	5·74
,, ,,	60·75	32·62 h + 34·05 k	32·88 h + 27·33 k	− 0·26 h + 6·72 k	5·16
,, ,,	60·72	35·05 h + 32·30 k	34·87 h + 26·75 k	0·18 h + 5·55 k	4·57
,, 3	59·85	39·93 h + 31·40 k	38·60 h + 26·73 k	1·33 h + 4·67 k	4·79
,, ,,	59·81	38·10 h + 33·60 k	38·80 h + 27·92 k	− 0·70 h + 5·68 k	3·98

Date.	Temp.	[B·E]	Y_{55}	Difference of Length in Micrometer Divisions.	Y_{55} − [B·E]
1865.	°				
June 27	63·80	55·22 h + 58·83 k	55·63 h + 51·05 k	− 0·41 h + 7·78 k	5·89
„ „	63·79	55·67 h + 58·67 k	51·67 h + 55·67 k	4·00 h + 3·00 k	5·58
„ „	63·77	49·92 h + 64·98 k	53·03 h + 55·95 k	− 3·11 h + 9·03 k	4·74
„ „	63·78	56·72 h + 57·35 k	52·60 h + 54·92 k	4·12 h + 2·43 k	5·22
„ „	63·78	55·68 h + 60·33 k	54·23 h + 53·60 k	1·45 h + 6·73 k	6·53
„ 28	63·50	55·25 h + 60·65 k	54·23 h + 53·92 k	1·02 h + 6·73 k	6·19
„ „	63·49	61·27 h + 53·92 k	54·23 h + 54·45 k	7·04 h − 0·53 k	5·18
„ „	63·47	57·65 h + 58·23 k	55·43 h + 53·70 k	2·22 h + 4·53 k	5·38
„ 29	63·30	62·97 h + 56·97 k	54·68 h + 57·93 k	8·29 h − 0·96 k	5·83
„ „	63·32	58·15 h + 62·45 k	57·05 h + 57·38 k	1·10 h + 5·07 k	4·92

Date.	Temp.	[C·F]	Y_{55}	Difference of Length in Micrometer Divisions.	Y_{55} − [C·F]
1865.	°				
May 31	61·21	25·60 h + 25·05 k	25·08 h + 23·93 k	0·52 h + 1·12 k	1·31
„ „	61·22	25·68 h + 20·65 k	25·42 h + 19·88 k	0·26 h + 0·77 k	0·82
„ „	61·28	20·52 h + 26·77 k	25·42 h + 21·92 k	− 4·90 h + 4·85 k	−0·02
„ „	61·35	19·13 h + 24·60 k	20·30 h + 22·50 k	− 1·17 h + 2·10 k	0·75
„ „	61·41	18·83 h + 29·18 k	10·88 h + 35·94 k	7·95 h − 6·76 k	0·93
„ „	61·41	21·92 h + 24·93 k	22·43 h + 24·58 k	− 0·51 h + 0·35 k	−0·13
June 1	60·79	21·43 h + 28·50 k	22·53 h + 28·03 k	− 1·10 h + 0·47 k	−0·50
„ „	60·74	21·38 h + 29·42 k	21·28 h + 29·48 k	0·10 h − 0·06 k	0·03
„ „	60·84	22·00 h + 30·38 k	28·52 h + 24·32 k	− 6·52 h + 6·06 k	−0·35
„ „	60·90	29·68 h + 21·90 k	26·85 h + 24·22 k	2·83 h − 2·32 k	0·40
„ 29	63·33	59·70 h + 56·05 k	58·60 h + 55·87 k	1·10 h + 0·18 k	1·02
„ „	63·43	54·20 h + 60·13 k	57·05 h + 56·50 k	− 2·85 h + 3·63 k	0·63
„ „	63·44	57·85 h + 59·10 k	58·98 h + 56·45 k	− 1·13 h + 2·65 k	1·22
„ „	63·38	58·73 h + 56·75 k	57·53 h + 56·73 k	1·20 h + 0·02 k	0·97
„ 30	63·15	58·55 h + 58·13 k	58·18 h + 57·42 k	0·37 h + 0·71 k	0·86
„ „	63·13	58·93 h + 57·80 k	58·08 h + 58·08 k	0·85 h − 0·28 k	0·45
„ „	63·09	61·58 h + 56·30 k	61·15 h + 56·17 k	0·43 h + 0·13 k	0·45
„ „	63·11	60·43 h + 55·83 k	62·33 h + 54·13 k	− 1·90 h + 1·70 k	−0·15
„ „	63·07	60·60 h + 57·48 k	59·65 h + 57·78 k	0·95 h − 0·30 k	0·52
„ „	63·08	55·07 h + 62·10 k	55·25 h + 60·67 k	− 0·18 h + 1·43 k	1·00

Date.	Temp.	[F·H]	Y_{55}	Difference of Length in Micrometer Divisions.	[F·H] − Y_{55}
1865.	°				
June 24	63·53	99·07 h + 98·37 k	126·68 h + 140·13 k	27·61 h + 41·76 k	55·32
„ „	63·61	99·55 h + 95·87 k	130·62 h + 135·15 k	31·07 h + 39·28 k	56·09
„ „	63·72	89·00 h + 106·13 k	130·02 h + 136·32 k	41·02 h + 30·19 k	56·75
„ „	63·82	92·88 h + 102·03 k	129·37 h + 135·87 k	36·49 h + 33·84 k	56·06
„ „	63·89	93·90 h + 101·01 k	132·00 h + 133·22 k	38·10 h + 32·21 k	56·04
„ 26	63·72	92·43 h + 101·63 k	127·23 h + 137·27 k	34·80 h + 35·64 k	56·15
„ „	63·77	96·68 h + 97·08 k	127·02 h + 136·45 k	30·34 h + 39·37 k	55·58
„ „	63·87	95·85 h + 96·43 k	127·98 h + 134·43 k	32·13 h + 38·00 k	55·92
„ „	63·99	86·81 h + 106·53 k	129·85 h + 132·06 k	43·04 h + 25·53 k	54·64
„ „	64·04	94·06 h + 98·10 k	130·85 h + 131·02 k	36·79 h + 32·92 k	55·57

In order to obtain from these comparisons—160 in number—the most probable values of the different intervals, let y be the excess of the expansion of a yard of $\mathbf{I}_6$ above that of $\mathbf{Y}_{55}$ and at 62° let

$$
\begin{aligned}
[\text{A}\cdot\text{B}] &= \mathbf{Y}_{55} + 30 + x_1 \\
[\text{A}\cdot\text{C}] &= \tfrac{4}{3}\,\mathbf{Y}_{55} + 15 + x_2 \\
[\text{A}\cdot\text{D}] &= \tfrac{5}{3}\,\mathbf{Y}_{55} + 5 + x_3 \\
[\text{A}\cdot\text{E}] &= \tfrac{6}{3}\,\mathbf{Y}_{55} + 25 + x_4 \\
[\text{A}\cdot\text{F}] &= \tfrac{7}{3}\,\mathbf{Y}_{55} + 15 + x_5 \\
[\text{A}\cdot\text{H}] &= \tfrac{10}{3}\,\mathbf{Y}_{55} + 70 + x_0
\end{aligned}
\qquad (1)
$$

Then we have equations of the following form:—

		No. of Equations.
$[\text{A}\cdot\text{B}] - \mathbf{Y}_{55}$	$= x_1 + (t - 62)\,y + 30$	10
$[\text{B}\cdot\text{E}] - \mathbf{Y}_{55}$	$= - x_1 + x_4 + (t - 62)\,y - 5$	30
$[\text{C}\cdot\text{F}] - \mathbf{Y}_{55}$	$= - x_2 + x_5 + (t - 62)\,y$	20
$[\text{F}\cdot\text{H}] - \mathbf{Y}_{55}$	$= - x_5 + x_0 + (t - 62)\,y + 55$	10
$[\text{A}\cdot\text{B}] - [\text{B}\cdot\text{E}]$	$= 2\,x_1 - x_4 + 35$	10
$[\text{B}\cdot\text{E}] - [\text{C}\cdot\text{F}]$	$= - x_1 + x_2 + x_4 - x_5 - 5$	20
$[\text{C}\cdot\text{F}] - [\text{F}\cdot\text{H}]$	$= - x_2 + 2\,x_5 - x_0 - 55$	10
$[\text{B}\cdot\text{D}] - [\text{C}\cdot\text{E}]$	$= - x_1 + x_3 + x_2 - x_4 - 35$	10
$[\text{C}\cdot\text{E}] - [\text{D}\cdot\text{F}]$	$= - x_2 + x_3 + x_4 - x_5$	10
$[\text{B}\cdot\text{D}] - [\text{D}\cdot\text{F}]$	$= - x_1 + 2\,x_3 - x_5 - 35$	10
$[\text{B}\cdot\text{C}] - [\text{C}\cdot\text{D}]$	$= - x_1 + 2\,x_2 - x_3 - 5$	10
$[\text{D}\cdot\text{E}] - [\text{E}\cdot\text{F}]$	$= - x_3 + 2\,x_4 - x_5 + 30$	10

(2)

The 160 equations multiplied through according to the method of least squares give finally— (3)

$$
\begin{aligned}
0 &= +670{\cdot}00 + 2080{\cdot}56y + 17{\cdot}96\,x_0 + 146{\cdot}42\,x_1 - 3{\cdot}36\,x_2 - 137{\cdot}27\,x_4 - 14{\cdot}60\,x_5 \\
0 &= -29{\cdot}07 + 17{\cdot}96y + 20{\cdot}0\,x_0 + 10\,x_2 - 30\,x_5 \\
0 &= +158{\cdot}34 + 146{\cdot}42y + 130\,x_1 - 70\,x_2 - 20\,x_3 - 60\,x_4 + 30\,x_5 \\
0 &= -100{\cdot}20 - 3{\cdot}36y + 10\,x_0 - 50\,x_1 + 110\,x_2 - 20\,x_3 - 50\,x_5 \\
0 &= -73{\cdot}97 - 20\,x_1 - 20\,x_2 + 80\,x_3 - 20\,x_4 - 20\,x_5 \\
0 &= -78{\cdot}84 - 137{\cdot}27y - 60\,x_1 - 20\,x_3 + 120\,x_4 - 50\,x_5 \\
0 &= +138{\cdot}46 - 14{\cdot}60y - 30\,x_0 + 30\,x_1 - 50\,x_2 - 20\,x_3 - 50\,x_4 + 120\,x_5
\end{aligned}
$$

By the solution of these equations we get

$$
\begin{aligned}
y &= -{\cdot}3009 \;\ldots\ldots\; \textit{weight} = 1740{\cdot}07 \\
x_0 &= +{\cdot}62 \;\ldots\ldots\; \text{,,} \quad = 4{\cdot}81 \\
x_1 &= -{\cdot}39 \\
x_2 &= +{\cdot}57 \\
x_3 &= +{\cdot}84 \\
x_4 &= +{\cdot}03 \\
x_5 &= -{\cdot}55
\end{aligned}
\qquad (4)
$$

If now we substitute these values of $y\ x_0\ x_1 \ldots \ldots x_5$ in the first seventy of equations (2), we get the following system of errors of the comparisons of the different yards on I_S with the yard Y_{55}:—

$[A \cdot B] - Y_{55}$	$[B \cdot E] - Y_{55}$			$[C \cdot F] - Y_{55}$		$[F \cdot H] - Y_{55}$
+ 0·58	+ 0·90	+ 0·45	+ 0·77	+ 0·43	− 0·50	+ 0·38
+ 0·11	− 2·13	+ 0·91	+ 0·46	− 0·07	− 0·92	− 0·41
− 0·06	+ 0·03	+ 0·53	+ 0·37	− 0·92	− 0·33	− 1·10
+ 1·16	+ 0·23	+ 0·46	+ 0·10	− 0·18	− 0·56	− 0·44
+ 0·78	− 0·06	+ 0·77	+ 1·41	− 0·01	− 0·60	− 0·44
+ 0·80	+ 0·42	+ 1·53	+ 1·16	− 1·07	− 1·01	− 0·50
+ 0·10	+ 0·13	+ 0·95	+ 0·15	− 1·26	− 1·00	+ 0·05
+ 0·28	+ 0·21	+ 0·37	+ 0·36	− 0·71	− 1·60	− 0·32
− 0·06	− 0·28	+ 0·85	+ 0·86	− 1·12	− 0·92	+ 0·93
− 0·03	+ 0·12	+ 0·06	− 0·06	− 0·39	− 0·44	− 0·02

The substitution of $y\ x_0 \ldots \ldots x_5$ in the remaining ninety equations of (2), gives the following system of errors of the comparisons of the different subdivisions of the bar one with another. The first four columns appertain to the yard spaces; the fifth, sixth, and seventh columns to the two-feet spaces; and the last two columns to one-foot spaces:—

[A·B] − [B·E]	[B·E] − [C·F]		[C·F] − [F·H]	[B·D] − [C·E]	[C·E] − [D·F]	[B·D] − [D·F]	[B·C] − [C·D]	[D·E] − [E·F]
− 1·20	+ 0·25	+ 1·14	+ 0·37	+ 0·72	− 0·92	+ 0·20	+ 0·10	− 1·77
− 0·11	+ 0·76	+ 0·65	+ 0·08	− 1·09	− 0·00	− 0·69	+ 0·84	− 2·59
− 0·34	+ 0·19	− 1·19	− 0·81	− 0·39	+ 0·26	+ 0·03	− 1·45	− 2·51
− 0·78	+ 0·09	− 0·73	+ 1·33	− 1·54	+ 0·57	+ 0·49	− 1·59	− 1·83
− 0·34	+ 0·50	− 0·29	+ 0·07	− 0·37	+ 0·37	− 0·46	+ 0·26	− 0·82
− 0·58	+ 1·06	− 1·35	+ 0·50	− 0·04	+ 0·09	− 1·21	− 0·78	− 0·20
+ 0·37	+ 0·81	− 1·29	− 0·09	− 1·03	− 0·11	− 0·11	− 0·17	− 0·22
− 0·82	+ 0·37	− 0·35	− 1·14	− 0·83	− 1·23	+ 0·05	− 0·12	− 0·11
+ 0·01	− 1·70	+ 0·26	− 1·27	− 1·12	+ 0·24	− 0·08	− 0·09	+ 0·34
+ 0·01	− 1·07	− 0·34	− 0·89	− 1·83	+ 0·20	− 0·62	− 0·01	− 0·16

Arranged as the errors are here, in groups, it appears that some groups are not wholly free from a suspicion of constant error. For instance, the two yards [B·E] [C·F] when compared with the Standard Yard Y_{55} do not exhibit the same difference of length as when compared directly one with the other.

The sum of the squares of the 160 errors = 103·97; hence the probable error of one comparison

$$= \cdot 674 \sqrt{\frac{103 \cdot 97}{160 - 7}} = \pm\ 0 \cdot 556 \qquad (5)$$

And the probable errors of x_0 and y—

$$x_0 \ldots\ldots \frac{\pm\ 0 \cdot 556}{\sqrt{4 \cdot 81}} = \pm\ 0 \cdot 25 \qquad (6)$$

$$y \ldots\ldots \frac{\pm\ 0 \cdot 556}{\sqrt{1740}} = \pm\ 0 \cdot 013 \qquad (7)$$

Combining equations (1) and (4) we get for the true lengths of the various intervals the following:—

$$
\begin{aligned}
[\text{A}\cdot\text{B}] &= \mathbf{Y}_{55} + 29\cdot61 \\
[\text{A}\cdot\text{C}] &= \tfrac{4}{3}\,\mathbf{Y}_{55} + 15\cdot57 \\
[\text{A}\cdot\text{D}] &= \tfrac{5}{3}\,\mathbf{Y}_{55} + 5\cdot84 \\
[\text{A}\cdot\text{E}] &= \tfrac{6}{3}\,\mathbf{Y}_{55} + 25\cdot03 \\
[\text{A}\cdot\text{F}] &= \tfrac{7}{3}\,\mathbf{Y}_{55} + 14\cdot45
\end{aligned}
\qquad (8)
$$

And for the length of the whole bar $[\text{A}\cdot\text{H}] = \mathbf{l}_{S}$

$$\mathbf{l}_{S} = \tfrac{10}{3}\,\mathbf{Y}_{55} + 70\cdot62 \pm 0\cdot25 \qquad (9)$$

XIX.

DETERMINATION OF THE LENGTH OF THE INDIAN STANDARD FOOT

AND ITS SUBDIVISIONS.

1.

This bar is of the same dimensions as **OF** described at page 14, but is of steel. The lines marking the inches and smaller subdivisions are drawn on gold pins let into the bar. The extreme inches are subdivided into twentieths. The 13 inch-lines are marked *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *k*, *l*, *m*, *n*, *p*.

This bar was compared with the scale **OF** (entire length) 10 times at about the temperature 53°, and 20 times at the mean temperature of about 64°·5.

The six-inch spaces [$a \cdot g$], [$d \cdot l$], [$g \cdot p$], were then compared amongst themselves 15 times; the five-inch spaces [$a \cdot f$], [$b \cdot g$], were compared together 10 times; the four-inch spaces [$a \cdot e$], [$b \cdot f$], [$c \cdot g$] were compared together 10 times; and finally the three-inch spaces [$a \cdot d$], [$b \cdot e$], [$c \cdot f$], [$d \cdot g$], were compared together 10 times. The comparisons of these intervals were conducted precisely as explained in the case of **OF**, Section IV.

The comparisons are given in the following Tables:—

Comparisons of the Ordnance Standard Foot with the Indian Standard Foot.

I.

Date. 1865.		Temp.	OF	IF	Difference of Length in Micrometer Divisions.	IF—OF
April	18	° 52·50	24·95 h + 25·15 k	22·20 h + 20·92 k	2·75 h + 4·23 k	5·57
"	"	52·66	23·40 h + 26·32 k	27·17 h + 15·97 k	−3·77 h + 10·35 k	5·27
"	"	52·87	23·92 h + 23·27 k	21·65 h + 21·55 k	2·27 h + 1·72 k	3·18
"	"	53·10	22·87 h + 23·92 k	23·52 h + 19·15 k	−0·65 h + 4·77 k	3·29
"	19	52·82	24·15 h + 28·00 k	21·40 h + 25·75 k	2·75 h + 2·25 k	3·98
"	20	52·88	25·15 h + 26·07 k	22·65 h + 22·97 k	2·50 h + 3·10 k	4·46
"	"	53·05	25·20 h + 25·20 k	21·70 h + 23·80 k	3·50 h + 1·40 k	3·90
"	"	53·08	20·95 h + 29·32 k	14·40 h + 29·60 k	6·55 h − 0·28 k	4·99
"	"	53·14	26·25 h + 24·07 k	22·45 h + 21·80 k	3·80 h + 2·27 k	4·84
"	"	53·19	24·77 h + 24·87 k	24·52 h + 19·70 k	0·25 h + 5·17 k	4·33
July	5	63·21	72·07 h + 77·50 k	71·15 h + 75·32 k	0·92 h + 2·18 k	2·47
"	"	63·36	70·85 h + 78·52 k	70·52 h + 75·07 k	0·33 h + 3·45 k	3·02
"	"	63·53	73·30 h + 76·27 k	70·97 h + 74·27 k	2·33 h + 2·00 k	3·45
"	"	63·68	68·87 h + 79·10 k	70·32 h + 74·30 k	−1·45 h + 4·80 k	2·68
"	6	63·85	68·20 h + 79·80 k	66·30 h + 78·25 k	1·90 h + 1·55 k	2·75
"	"	64·04	68·15 h + 79·60 k	69·15 h + 74·45 k	−1·00 h + 5·15 k	3·32
"	"	64·24	68·17 h + 78·95 k	69·15 h + 73·55 k	−0·98 h + 5·40 k	3·53
"	"	64·42	69·10 h + 79·65 k	68·65 h + 74·65 k	0·45 h + 5·00 k	4·35
"	"	64·55	69·47 h + 77·55 k	69·07 h + 74·57 k	0·40 h + 2·98 k	2·70
"	7	64·73	70·42 h + 77·95 k	69·40 h + 75·42 k	1·02 h + 2·53 k	2·83
"	"	64·77	71·92 h + 75·77 k	72·47 h + 70·40 k	−0·55 h + 5·37 k	3·85
"	"	64·86	75·27 h + 71·75 k	74·05 h + 70·25 k	1·22 h + 1·50 k	2·17
"	"	64·96	71·72 h + 75·40 k	67·90 h + 75·52 k	3·82 h − 0·12 k	2·94
"	"	65·12	72·52 h + 73·00 k	71·20 h + 71·05 k	1·32 h + 1·95 k	2·61
"	"	65·22	70·27 h + 76·37 k	69·47 h + 73·35 k	0·80 h + 3·02 k	3·05
"	8	65·01	73·72 h + 73·27 k	72·25 h + 71·50 k	1·47 h + 1·77 k	2·58
"	"	65·11	71·75 h + 74·82 k	71·65 h + 71·32 k	0·10 h + 3·50 k	2·87
"	"	65·21	74·30 h + 73·10 k	71·82 h + 71·55 k	2·48 h + 1·55 k	3·21
"	"	65·33	73·70 h + 73·05 k	71·80 h + 71·67 k	1·90 h + 1·38 k	2·61
"	"	65·36	64·17 h + 83·52 k	63·32 h + 81·60 k	0·85 h + 1·92 k	2·21

Comparisons of Subdivisions.

II.

Z—[a·g]	Z—[d·l]	Z—[g·p]
14·85 h + 15·90 k	15·85 h + 14·17 k	15·07 h + 10·57 k
15·80 h + 15·67 k	16·32 h + 14·50 k	17·02 h + 8·77 k
15·82 h + 16·42 k	15·17 h + 14·47 k	15·15 h + 11·45 k
11·97 h + 15·97 k	16·40 h + 11·98 k	16·15 h + 10·42 k
15·92 h + 13·77 k	13·50 h + 15·85 k	12·50 h + 12·33 k
14·55 h + 14·85 k	15·75 h + 13·90 k	19·07 h + 6·85 k
14·25 h + 16·02 k	15·02 h + 15·12 k	17·07 h + 8·62 k
14·45 h + 15·92 k	15·75 h + 15·35 k	15·60 h + 10·72 k
12·60 h + 16·78 k	13·87 h + 16·40 k	13·17 h + 13·15 k
14·80 h + 15·37 k	16·40 h + 13·82 k	16·12 h + 9·67 k
15·10 h + 10·45 k	12·52 h + 12·20 k	12·35 h + 9·10 k
11·00 h + 14·47 k	9·72 h + 14·25 k	8·90 h + 12·12 k
12·52 h + 12·72 k	14·10 h + 10·87 k	11·20 h + 10·07 k
14·80 h + 9·87 k	13·77 h + 11·72 k	16·10 h + 5·20 k
15·10 h + 10·30 k	13·20 h + 12·77 k	9·97 h + 11·50 k

III.

Z — [*a*·*f*]	Z — [*b*·*g*]
14·17 *h* + 13·35 *k*	15·40 *h* + 10·15 *k*
12·17 *h* + 15·35 *k*	11·77 *h* + 14·37 *k*
12·85 *h* + 15·20 *k*	13·77 *h* + 11·45 *k*
13·77 *h* + 14·17 *k*	16·42 *h* + 9·12 *k*
12·32 *h* + 14·75 *k*	11·10 *h* + 14·17 *k*
12·22 *h* + 14·05 *k*	11·87 *h* + 13·32 *k*
11·75 *h* + 14·70 *k*	10·82 *h* + 14·37 *k*
8·70 *h* + 11·97 *k*	7·90 *h* + 10·65 *k*
8·50 *h* + 11·25 *k*	8·32 *h* + 10·15 *k*
7·62 *h* + 13·05 *k*	6·90 *h* + 13·35 *k*

IV.

Z — [*a*·*e*]	Z — [*b*·*f*]	Z — [*c*·*g*]
8·82 *h* + 13·60 *k*	11·52 *h* + 8·95 *k*	13·40 *h* + 10·62 *k*
9·62 *h* + 13·67 *k*	11·42 *h* + 9·27 *k*	11·70 *h* + 12·82 *k*
11·35 *h* + 11·50 *k*	11·70 *h* + 9·17 *k*	9·97 *h* + 11·67 *k*
10·02 *h* + 12·85 *k*	9·80 *h* + 11·05 *k*	10·70 *h* + 12·55 *k*
9·47 *h* + 13·10 *k*	9·45 *h* + 10·32 *k*	10·05 *h* + 13·27 *k*
13·02 *h* + 9·52 *k*	10·40 *h* + 11·37 *k*	10·02 *h* + 13·15 *k*
12·02 *h* + 11·50 *k*	10·25 *h* + 11·15 *k*	9·15 *h* + 14·45 *k*
12·57 *h* + 11·10 *k*	13·35 *h* + 8·72 *k*	13·52 *h* + 11·00 *k*
10·77 *h* + 12·32 *k*	8·72 *h* + 12·90 *k*	7·57 *h* + 14·50 *k*
10·77 *h* + 11·90 *k*	11·40 *h* + 10·15 *k*	11·60 *h* + 12·10 *k*

V.

Z — [*a*·*d*]	Z — [*b*·*e*]	Z — [*c*·*f*]	Z — [*d*·*g*]
14·35 *h* + 13·87 *k*	13·12 *h* + 14·05 *k*	14·75 *h* + 14·57 *k*	15·65 *h* + 14·62 *k*
15·17 *h* + 13·22 *k*	14·20 *h* + 13·47 *k*	15·25 *h* + 15·30 *k*	15·00 *h* + 16·87 *k*
12·70 *h* + 16·47 *k*	12·17 *h* + 16·50 *k*	12·97 *h* + 17·30 *k*	13·57 *h* + 17·60 *k*
11·97 *h* + 17·15 *k*	14·60 *h* + 13·90 *k*	16·22 *h* + 13·00 *k*	15·72 *h* + 16·00 *k*
13·17 *h* + 15·35 *k*	14·95 *h* + 13·67 *k*	15·32 *h* + 14·05 *k*	15·92 *h* + 14·00 *k*
14·35 *h* + 15·07 *k*	13·77 *h* + 15·25 *k*	13·00 *h* + 16·50 *k*	14·80 *h* + 16·97 *k*
15·12 *h* + 14·05 *k*	14·35 *h* + 14·57 *k*	15·60 *h* + 14·65 *k*	17·37 *h* + 14·37 *k*
15·37 *h* + 12·62 *k*	14·52 *h* + 14·17 *k*	15·35 *h* + 15·47 *k*	15·55 *h* + 15·72 *k*
13·38 *h* + 15·38 *k*	12·45 *h* + 15·45 *k*	14·23 *h* + 16·33 *k*	13·85 *h* + 17·40 *k*
13·28 *h* + 15·43 *k*	12·05 *h* + 15·93 *k*	14·63 *h* + 15·58 *k*	16·78 *h* + 15·03 *k*

Substituting the values of *h* and *k* in the last four Tables, they become—

VI.

Z − [*a*·*g*]	Z − [*d*·*l*]	Z − [*g*·*p*]
24·49	23·91	20·41
25·06	24·54	20·53
25·68	23·60	21·18
22·26	22·60	21·15
23·64	23·38	19·77
23·42	23·61	20·62
24·11	24·00	20·45
24·19	24·77	20·95
23·41	24·11	20·96
24·03	24·06	20·53
20·34	19·69	17·08
20·29	19·10	16·75
20·10	19·88	16·94
19·64	20·30	16·95
20·22	20·68	17·10

VII.

Z − [*a*·*f*]	Z − [*b*·*g*]
21·92	20·34
21·92	20·82
22·34	20·08
22·25	20·33
21·56	20·13
20·93	20·06
21·07	20·07
16·47	14·78
15·73	14·71
16·47	16·14

VIII.

Z − [*a*·*e*]	Z − [*b*·*f*]	Z − [*c*·*g*]
17·86	16·30	19·13
18·56	16·47	19·53
18·20	16·62	17·24
18·22	16·61	18·52
17·98	15·75	18·58
17·95	17·34	18·46
18·73	17·05	18·80
18·85	17·57	19·52
18·39	17·22	17·59
18·06	17·16	18·88

IX.

$Z - [a \cdot d]$	$Z - [b \cdot e]$	$Z - [c \cdot f]$	$Z - [d \cdot g]$
22·47	21·64	23·35	24·11
22·61	22·04	24·33	25·38
23·24	22·84	24·11	24·83
23·20	22·70	23·27	25·26
22·72	22·79	23·39	23·83
23·43	23·11	23·50	25·31
23·23	23·03	24·09	25·27
22·29	22·85	24·55	24·90
22·91	22·23	24·34	24·89
22·87	22·29	24·06	25·33

2.

Let the excess of the scale **IF** above **OF** be expressed by

$$\mathbf{IF} - \mathbf{OF} = x + y\,(t - 62) \qquad (1)$$

x being the difference of length at 62°, and y the difference of expansion of the two bars for 1° Fahrenheit; then the first of the above tables supplies us with 30 equations, which, treated by the method of least squares, resolve themselves into

$$\begin{aligned} 30\,x - 40\cdot15\ y - 103\cdot01 &= 0 \\ -40\cdot15\,x + 960\cdot306\,y + 250\cdot73 &= 0. \end{aligned} \qquad (2)$$

If we write A and B for the absolute terms

$$\begin{aligned} x + 0\cdot035309\ \mathrm{A} + 0\cdot001476\ \mathrm{B} &= 0 \\ y + 0\cdot001476\ \mathrm{A} + 0\cdot001103\ \mathrm{B} &= 0, \end{aligned} \qquad (3)$$

which give us finally for x and y,

$$\begin{aligned} x &= +\ 3\cdot27 \ \ldots\ldots\ldots\ \textit{Reciprocal of weight} = \cdot0353 \\ y &= -\ 0\cdot1245 \ \ldots\ldots\ldots\ \quad ,, \quad ,, \quad = \cdot0011 \end{aligned} \qquad (4)$$

These values substituted in the equations of condition give the following system of errors:—

April 18, 19, 20.	July 5, 6.	July 7, 8.
−1·12	+0·65	−0·93
−0·84	+0·08	+0·74
+1·22	−0·37	+0·04
+1·07	+0·38	+0·27
+0·43	+0·29	−0·18
−0·06	−0·31	+0·31
+0·48	−0·54	+0·01
−0·61	−1·39	−0·34
−0·47	+0·25	+0·25
+0·03	+0·10	+0·64

The sum of the squares of these errors is 10·996; hence the probable error of a single comparison is

$$\pm 0 \cdot 674 \sqrt{\frac{10 \cdot 996}{30 - 2}} = \pm 0 \cdot 422 \qquad (5)$$

and the probable errors of x and y

$$x \ldots\ldots \pm 0 \cdot 422 \sqrt{\cdot 0353} = \pm 0 \cdot 079 \qquad (6)$$
$$y \ldots\ldots \pm 0 \cdot 422 \sqrt{\cdot 0011} = \pm 0 \cdot 014$$

We have then, at 62°, the difference in length of the two scales

$$\mathbf{IF} - \mathbf{OF} = + 3 \cdot 27 \pm 0 \cdot 079; \qquad (7)$$

but by equation (7) page 77, the length of **OF** is

$$\tfrac{1}{3}\,\mathbf{Y}_{55} - 0 \cdot 36 \pm 0 \cdot 108;$$

consequently, the length of the Indian Standard Foot at 62° is

$$\tfrac{1}{3}\,\mathbf{Y}_{55} + 2 \cdot 91 \pm 0 \cdot 134. \qquad (8)$$

3.

Let the values of the spaces $[a \cdot g]$, $[g \cdot p]$, $[d \cdot l]$ be as follows:—

$$[a \cdot g] = \tfrac{1}{2}\,[a \cdot p] + x_g \qquad (9)$$
$$[g \cdot p] = \tfrac{1}{2}\,[a \cdot p] - x_g$$
$$[d \cdot l] = \tfrac{1}{2}\,[a \cdot p] + x$$

And referring to Table VI., let $\frac{1}{2}\,[a \cdot p] - z_n$ be the distance of the microscopes in the nth comparison or line, while the quantities entered in that line are $\alpha_n\ \beta_n\ \gamma_n$; then we have the following:—

$$\begin{array}{lll} x_g + z_1 + \alpha_1 = 0 & x + z_1 + \beta_1 = 0 & -x_g + z_1 + \gamma_1 = 0 \\ x_g + z_2 + \alpha_2 = 0 & x + z_2 + \beta_2 = 0 & -x_g + z_2 + \gamma_2 = 0 \\ \vdots & \vdots & \vdots \\ x_g + z_{15} + \alpha_{15} = 0 & x + z_{15} + \beta_{15} = 0 & -x_g + z_{15} + \gamma_{15} = 0 \end{array} \qquad (10)$$

in all, 45 equations. These solved give—

$$30\,x - (\alpha) + 2\,(\beta) - (\gamma) = 0 \qquad (11)$$
$$30\,x_g + (\alpha) \qquad\qquad - (\gamma) = 0$$

whence

$$x_g = -1 \cdot 65 \ldots\ldots \textit{weight} = 30 \qquad (12)$$
$$x = -1 \cdot 47 \ldots\ldots \quad ,, \quad = 10$$

The sum of the squares of the errors of the equations is 8·279; hence the probable error of an equation is

$$\pm 0{\cdot}674 \sqrt{\frac{8{\cdot}279}{45 - 17}} = \pm 0{\cdot}366 \qquad (13)$$

and the probable errors of x_g and x

$$x_g \ldots\ldots\ldots \frac{\pm {\cdot}366}{\sqrt{30}} = \pm {\cdot}067 \qquad (14)$$

$$x \ldots\ldots\ldots \frac{\pm {\cdot}366}{\sqrt{10}} = \pm {\cdot}116$$

We have then for $[a.g]$, $[g{\cdot}p]$, and $[d{\cdot}l]$ in terms of $[a{\cdot}p]$ the following,—

$$\begin{aligned} [a{\cdot}g] &= \tfrac{1}{2}[a{\cdot}p] - 1{\cdot}65 \pm 0{\cdot}067 \\ [g{\cdot}p] &= \tfrac{1}{2}[a{\cdot}p] + 1{\cdot}65 \pm 0{\cdot}067 \\ [d{\cdot}l] &= \tfrac{1}{2}[a{\cdot}p] - 1{\cdot}47 \pm 0{\cdot}116 \end{aligned} \qquad (15)$$

Again, to determine the errors of the lines b, c, d, e, f, with respect to a and g, let

$$\begin{aligned} [a{\cdot}b] &= \tfrac{1}{6}[a{\cdot}g] + x_b \\ [a{\cdot}c] &= \tfrac{2}{6}[a{\cdot}g] + x_c \\ [a{\cdot}d] &= \tfrac{3}{6}[a{\cdot}g] + x_d \\ [a{\cdot}e] &= \tfrac{4}{6}[a{\cdot}g] + x_e \\ [a{\cdot}f] &= \tfrac{5}{6}[a{\cdot}g] + x_f \end{aligned} \qquad (16)$$

and proceeding as at page 48, we find ultimately from the numbers in Tables VII, VIII, and IX,

$$\begin{aligned} 90\,(x_b + x_f) + 109{\cdot}26 &= 0 \\ 90\,(x_b - x_f) + 139{\cdot}32 &= 0 \\ 90\,(x_c - x_e) - 17{\cdot}19 &= 0 \\ 90\,(x_c + x_e) - 21{\cdot}51 &= 0 \\ 20\,x_d - 20{\cdot}14 &= 0 \end{aligned} \qquad (17)$$

whence

$$\begin{aligned} x_b &= -\,1{\cdot}38 \ldots\ldots\ \textit{weight} = \tfrac{180}{11} \\ x_c &= +\,0{\cdot}21 \ldots\ldots\ \text{,,} \quad = \tfrac{180}{17} \\ x_d &= +\,1{\cdot}01 \ldots\ldots\ \text{,,} \quad = \tfrac{180}{9} \\ x_e &= +\,0{\cdot}02 \ldots\ldots\ \text{,,} \quad = \tfrac{180}{17} \\ x_f &= +\,0{\cdot}17 \ldots\ldots\ \text{,,} \quad = \tfrac{180}{11} \end{aligned} \qquad (18)$$

Finding next the values of the z's, and substituting in the equations of condition, we find the errors and the sum of their squares equal to 11·1015. Hence the probable error of an equation is

$$\pm 0{\cdot}674 \sqrt{\frac{11{\cdot}1015}{90 - 35}} = \pm 0{\cdot}303 \qquad (19)$$

whence we have finally—

$$\begin{aligned} [a{\cdot}b] &= \tfrac{1}{6}[a{\cdot}g] - 1{\cdot}38 \pm 0{\cdot}074 \\ [a{\cdot}c] &= \tfrac{2}{6}[a{\cdot}g] + 0{\cdot}21 \pm 0{\cdot}093 \\ [a{\cdot}d] &= \tfrac{3}{6}[a{\cdot}g] + 1{\cdot}01 \pm 0{\cdot}068 \\ [a{\cdot}e] &= \tfrac{4}{6}[a{\cdot}g] + 0{\cdot}02 \pm 0{\cdot}093 \\ [a{\cdot}f] &= \tfrac{5}{6}[a{\cdot}g] + 0{\cdot}17 \pm 0{\cdot}074 \end{aligned} \qquad (20)$$

To obtain these intervals in terms of $[a \cdot g]$ we must make use of the first of equations (15)—

$$
\begin{aligned}
[a \cdot b] &= \tfrac{1}{12}\,[a \cdot p] - 1 \cdot 65 \pm \cdot 075 \\
[a \cdot c] &= \tfrac{2}{12}\,[a \cdot p] - 0 \cdot 34 \pm \cdot 096 \\
[a \cdot d] &= \tfrac{3}{12}\,[a \cdot p] + 0 \cdot 18 \pm \cdot 076 \\
[a \cdot e] &= \tfrac{4}{12}\,[a \cdot p] - 1 \cdot 08 \pm \cdot 103 \\
[a \cdot f] &= \tfrac{5}{12}\,[a \cdot p] - 1 \cdot 20 \pm \cdot 093
\end{aligned}
\qquad (21)
$$

But we have seen that at the temperature 62° Fahrenheit—

$$[a \cdot p] = \tfrac{1}{3}\,\mathbf{Y}_{55} + 2 \cdot 91 \pm 0 \cdot 134$$

and substituting this in equations (21) and (15)—

$$
\begin{aligned}
[a \cdot b] &= \tfrac{1}{36}\,\mathbf{Y}_{55} - 1 \cdot 41 \pm \cdot 076 \\
[a \cdot c] &= \tfrac{2}{36}\,\mathbf{Y}_{55} + 0 \cdot 14 \pm \cdot 098 \\
[a \cdot d] &= \tfrac{3}{36}\,\mathbf{Y}_{55} + 0 \cdot 91 \pm \cdot 083 \\
[a \cdot e] &= \tfrac{4}{36}\,\mathbf{Y}_{55} - 0 \cdot 11 \pm \cdot 112 \\
[a \cdot f] &= \tfrac{5}{36}\,\mathbf{Y}_{55} + 0 \cdot 01 \pm \cdot 108 \\
[a \cdot g] &= \tfrac{6}{36}\,\mathbf{Y}_{55} - 0 \cdot 19 \pm \cdot 095 \\
[d \cdot l] &= \tfrac{6}{36}\,\mathbf{Y}_{55} - 0 \cdot 01 \pm \cdot 134
\end{aligned}
\qquad (22)
$$

These are the definite values of the different spaces. The errors are very small, and the central six-inch space on the scale $[d \cdot l]$, is doubtless the most precise ever laid off.

XX.

COMPARISONS OF

THE OLD AND NEW INDIAN STANDARDS

AND

O_1

1.

THE old Indian Standard **B**, or, as it is designated in the "*Account of the Measurement of the Lough Foyle Base*," I_b, is a bar similar in general construction to the Ordnance Survey Standard O_1, but is not so strong: it is barely an inch in breadth, and only two inches in depth—supported on rollers at one-fourth and three-fourths of its length: it is painted white. The dots are in excellent preservation.

This bar was compared with O_2 (a bar similar in every respect to O_1) in 1831, by the late Lieutenant Murphy, R.E.; and again, in 1846, it was compared at Southampton with O_1. Taking into account the difference of O_1 and O_2, the results are as follows:—(See "*Account of the Measurement of the Lough Foyle Base*," page 96.)

$$1831 \ldots\ldots\ldots\ldots I_b = O_1 - \cdot 000801 \text{ inch.}$$
$$1846 \ldots\ldots\ldots\ldots I_b = O_1 - \cdot 000865 \text{ ,,}$$

or, using the millionth of the yard as unit of small quantities,

$$\begin{aligned} 1831 \ldots\ldots\ldots\ldots I_b &= O_1 - 22\cdot 25 \\ 1846 \ldots\ldots\ldots\ldots I_b &= O_1 - 24\cdot 03 \end{aligned} \tag{1}$$

No direct experiments on the value of the expansion of I_b have ever been made, so that there is some uncertainty on this point. We assume it in the comparisons now to be recorded, as equal to the expansion of O_1.

With respect to the expansion of O_1, it appears from page 93 that it is less than the expansion of OI_1 by $0\cdot 0446 \pm \cdot 0236$. Now the direct experiments on the latter bar give for its expansion under 1° Fahrenheit $21\cdot 5763 \pm \cdot 0103$. Hence the expansion of O_1 is $21\cdot 532 \pm \cdot 026$. For I_S and I_B we take the results of the second series of experiments, viz., $21\cdot 159 \pm \cdot 019$ and $32\cdot 759 \pm \cdot 019$.—(See page 216.)

I_S was compared with O_1 twenty-one times; I_B with O_1 twenty times; I_S was compared with I_b ten times; and I_B with I_b thirteen times. Of these last, one comparison appears to be in error just nine micrometer divisions. As this is a case without a parallel, it is assumed that one of the micrometers was read ten divisions in error, and this comparison is rejected in getting the mean result.

Tables I., II., III., IV., contain the comparisons, with the correction for temperature to reduce the differences of length to what they would be at 62°.

I.

Date.	Temp.	l_b	l_S	Difference of Length in Micrometer Divisions.	Equivalents in Millionths of a Yard.	Correction for Temp.	$l_S - l_b$
1865.	°						
June 5	60·16	123·68 h + 129·12 k	67·72 h + 74·98 k	55·96 h + 54·14 k	87·77	−0·68	87·09
„ „	60·57	117·63 h + 128·10 k	72·87 h + 63·93 k	44·76 h + 64·17 k	86·87	−0·53	86·34
„ „	60·68	119·42 h + 122·97 k	68·47 h + 67·15 k	50·95 h + 55·82 k	85·12	−0·49	84·63
„ 6	60·95	123·30 h + 117·83 k	69·45 h + 59·98 k	53·85 h + 57·85 k	89·05	−0·39	88·66
„ „	61·12	119·48 h + 120·18 k	67·97 h + 58·73 k	51·51 h + 61·45 k	90·07	−0·33	89·74
„ „	61·30	115·42 h + 118·77 k	69·30 h + 56·28 k	46·12 h + 62·49 k	86·61	−0·26	86·35
„ „	61·44	118·55 h + 116·13 k	61·52 h + 63·88 k	57·03 h + 52·25 k	87·11	−0·21	86·90
„ „	61·64	121·68 h + 106·05 k	59·23 h + 64·51 k	62·45 h + 41·54 k	82·87	−0·13	82·74
„ 7	62·26	101·95 h + 116·40 k	44·25 h + 65·33 k	57·70 h + 51·07 k	86·70	+0·10	86·80
„ „	62·38	103·47 h + 113·42 k	52·92 h + 56·58 k	50·55 h + 56·84 k	85·62	+0·14	85·76

Mean Temperature 61°·25.

II.

Date.	Temp.	l_b	l_B	Difference of Length in Micrometer Divisions.	Equivalents in Millionths of a Yard.	Correction for Temp.	$l_B - l_b$
1865.	°						
June 8	62·68	243·37 h + 234·03 k	87·10 h + 104·45 k	156·27 h + 129·58 k	227·84	− 7·63	220·21
„ „	62·74	240·00 h + 236·00 k	87·50 h + 103·87 k	152·50 h + 132·13 k	226·87	− 8·31	218·56
„ „	62·81	235·18 h + 241·70 k	98·43 h + 90·68 k	136·75 h + 151·02 k	229·43	− 9·09	220·34
„ „	62·89	237·42 h + 237·22 k	102·73 h + 86·13 k	134·69 h + 151·09 k	227·85	− 9·99	217·86
„ „	62·93	242·85 h + 231·13 k	86·90 h + 99·47 k	155·95 h + 131·66 k	229·24	−10·44	218·80
„ 9	63·06	243·62 h + 228·43 k	94·17 h + 89·32 k	149·45 h + 139·11 k	230·02	−11·90	218·12
„ „	63·11	236·10 h + 234·38 k	92·37 h + 89·37 k	143·73 h + 145·01 k	230·18	−12·46	217·72
„ „	63·19	229·97 h + 241·80 k	87·98 h + 93·47 k	141·99 h + 148·33 k	231·45	−13·36	218·09
„ „	63·29	225·08 h + 245·93 k	92·97 h + 87·48 k	132·11 h + 158·45 k	231·67	−14·48	217·19
„ „	63·41	232·75 h + 237·78 k	92·81 h + 85·20 k	139·94 h + 152·58 k	233·21	−15·83	217·38
„ 10	63·64	241·77 h + 226·82 k	83·63 h + 85·62 k	158·14 h + 141·20 k	238·60	−18·41	220·19
„ „	63·76	236·38 h + 230·22 k	89·10 h + 87·40 k	147·28 h + 142·82 k	231·26	−19·76	211·50
„ „	63·84	234·53 h + 231·75 k	78·43 h + 87·75 k	156·10 h + 144·00 k	239·22	−20·66	218·56

Mean Temperature 63°·18.

III.

Date.	Temp.	O_1	I_S	Difference of Length in Micrometer Divisions.	Equivalents in Millionths of a Yard.	Correction for Temp.	$I_S - O_1$
1865.	°						
June 14	64·33	193·93 *h* + 201·25 *k*	169·18 *h* + 149·55 *k*	24·75 *h* + 51·70 *k*	60·99	+0·87	61·86
„ „	64·28	195·35 *h* + 202·38 *k*	158·92 *h* + 162·62 *k*	36·43 *h* + 39·76 *k*	60·74	+0·85	61·59
„ 15	64·07	202·45 *h* + 198·83 *k*	165·40 *h* + 161·48 *k*	37·05 *h* + 37·35 *k*	59·31	+0·77	60·08
„ „	64·13	200·37 *h* + 201·65 *k*	165·23 *h* + 159·98 *k*	35·14 *h* + 41·67 *k*	61·24	+0·79	62·03
„ „	64·17	200·38 *h* + 198·70 *k*	167·92 *h* + 156·10 *k*	32·46 *h* + 42·60 *k*	59·85	+0·81	60·66
„ „	64·20	203·90 *h* + 198·45 *k*	161·08 *h* + 162·58 *k*	42·82 *h* + 35·87 *k*	62·72	+0·82	63·54
„ 16	64·13	199·70 *h* + 206·08 *k*	162·92 *h* + 163·08 *k*	36·78 *h* + 43·00 *k*	63·61	+0·79	64·40
„ „	64·13	194·75 *h* + 211·22 *k*	164·48 *h* + 161·58 *k*	30·27 *h* + 49·64 *k*	63·73	+0·79	64·52
„ „	64·09	198·08 *h* + 204·35 *k*	160·93 *h* + 163·92 *k*	37·15 *h* + 40·43 *k*	61·85	+0·78	62·63
„ „	64·11	197·83 *h* + 205·43 *k*	163·40 *h* + 161·32 *k*	34·43 *h* + 44·11 *k*	62·63	+0·78	63·41
„ 17	63·77	213·93 *h* + 203·53 *k*	170·50 *h* + 168·35 *k*	43·43 *h* + 35·18 *k*	62·65	+0·66	63·31
„ „	63·68	207·12 *h* + 214·52 *k*	171·82 *h* + 168·78 *k*	35·30 *h* + 45·74 *k*	64·62	+0·62	65·24
„ „	63·61	209·57 *h* + 212·63 *k*	164·35 *h* + 179·12 *k*	45·22 *h* + 33·51 *k*	62·74	+0·60	63·34
„ „	63·59	212·70 *h* + 213·82 *k*	165·82 *h* + 180·10 *k*	46·88 *h* + 33·72 *k*	64·23	+0·59	64·82
„ „	63·55	202·77 *h* + 219·92 *k*	164·62 *h* + 180·77 *k*	38·15 *h* + 39·15 *k*	61·62	+0·58	62·20
„ „	63·46	206·38 *h* + 221·80 *k*	168·78 *h* + 179·82 *k*	37·60 *h* + 41·98 *k*	63·45	+0·54	63·99
„ 19	62·35	219·52 *h* + 231·42 *k*	190·45 *h* + 176·07 *k*	29·07 *h* + 55·35 *k*	67·34	+0·13	67·47
„ „	62·35	216·63 *h* + 226·03 *k*	192·75 *h* + 172·82 *k*	23·88 *h* + 53·21 *k*	61·50	+0·13	61·63
„ „	62·34	221·60 *h* + 224·48 *k*	186·28 *h* + 179·70 *k*	35·32 *h* + 44·78 *k*	63·87	+0.13	64·00
„ „	62·31	226·10 *h* + 219·20 *k*	183·00 *h* + 184·30 *k*	43·10 *h* + 34·90 *k*	62·17	+0·12	62·29
„ „	62·19	230·88 *h* + 222·92 *k*	184·27 *h* + 187·00 *k*	46·61 *h* + 35·92 *k*	65·78	+0·07	65·85

Mean Temperature 63°·56.

IV.

Date.	Temp.	O_1	I_B	Difference of Length in Micrometer Divisions.	Equivalents in Millionths of a Yard.	Correction for Temp.	$I_B - O_1$
1865.	°						
June 12	64·51	264·58 *h* + 261·23 *k*	126·03 *h* + 116·12 *k*	138·55 *h* + 145·11 *k*	226·14	−28·18	197·96
„ „	64·49	264·82 *h* + 263·63 *k*	128·32 *h* + 116·62 *k*	136·50 *h* + 147·01 *k*	226·03	−27·96	198·07
„ „	64·51	261·50 *h* + 265·95 *k*	120·00 *h* + 130·75 *k*	141·50 *h* + 135·20 *k*	220·57	−28·18	192·39
„ „	64·48	254·12 *h* + 273·87 *k*	124·77 *h* + 123·28 *k*	129·35 *h* + 150·59 *k*	223·19	−27·84	195·35
„ „	64·50	260·22 *h* + 267·43 *k*	119·63 *h* + 129·58 *k*	140·59 *h* + 137·85 *k*	221·95	−28·07	193·88
„ „	64·44	264·12 *h* + 265·98 *k*	127·28 *h* + 122·30 *k*	136·84 *h* + 143·68 *k*	223·64	−27·39	196·25
„ 13	64·17	267·15 *h* + 269·60 *k*	128·13 *h* + 130·57 *k*	139·02 *h* + 139·03 *k*	221·66	−24·36	197·30
„ „	64·07	266·95 *h* + 273·13 *k*	132·88 *h* + 130·30 *k*	134·07 *h* + 142·83 *k*	220·75	−23·24	197·51
„ „	64·04	263·50 *h* + 275·95 *k*	126·92 *h* + 139·12 *k*	136·58 *h* + 136·83 *k*	217·96	−22·90	195·06
„ „	64·04	265·88 *h* + 274·44 *k*	130·12 *h* + 134·78 *k*	135·76 *h* + 139·66 *k*	219·57	−22·90	196·67
„ 20	60·67	282·60 *h* + 290·68 *k*	169·73 *h* + 174·48 *k*	112·87 *h* + 116·20 *k*	182·61	+14·93	197·54
„ „	60·65	283·53 *h* + 289·58 *k*	167·98 *h* + 179·42 *k*	115·55 *h* + 110·16 *k*	179·92	+15·16	195·08
„ „	60·72	290·70 *h* + 280·53 *k*	167·77 *h* + 178·30 *k*	122·93 *h* + 102·23 *k*	179·46	+14·37	193·83
„ „	60·81	283·82 *h* + 284·82 *k*	167·87 *h* + 174·18 *k*	115·95 *h* + 110·64 *k*	180·62	+13·36	193·98
„ „	60·99	281·37 *h* + 286·95 *k*	166·73 *h* + 170·87 *k*	114·64 *h* + 116·08 *k*	183·93	+11·34	195·27
„ „	61·16	280·88 *h* + 281·77 *k*	165·90 *h* + 163·52 *k*	114·98 *h* + 118·25 *k*	185·93	+ 9·43	195·36
„ 21	61·41	274·03 *h* + 279·53 *k*	160·52 *h* + 158·35 *k*	113·51 *h* + 121·18 *k*	187·10	+ 6·62	193·72
„ „	61·49	274·50 *h* + 275·70 *k*	155·92 *h* + 159·07 *k*	118·58 *h* + 116·63 *k*	187·50	+ 5·73	193·23
„ „	61·60	271·53 *h* + 281·53 *k*	155·40 *h* + 158·50 *k*	116·13 *h* + 123·03 *k*	190·66	+ 4·49	195·15
„ „	61·68	274.40 *h* + 277·25 *k*	153·97 *h* + 159·42 *k*	120·43 *h* + 117·83 *k*	189·93	+ 3·59	193·52

Mean Temperature 62°·72.

2.

The mean of the quantities in the last column of Table I. is 86·50, leaving the following errors:—

Date.	Error.
1865.	
June 5	+ 0·59
" "	− 0·16
" "	+ 1·87
" 6	+ 2·16
" "	+ 3·24
" "	− 0·15
" "	+ 0·40
" "	− 3·76
" 7	+ 0·30
" "	− 0·74

the sum of the squares of which is 33·99. Hence the probable error of the mean is

$$\pm 0{\cdot}674\sqrt{\frac{33{\cdot}99}{90}} = \pm 0{\cdot}415$$

and therefore

$$I_S - I_b = 86{\cdot}50 \pm 0{\cdot}41 \tag{2}$$

The mean of the quantities in the last column of Table II. is (omitting the last comparison but one, for the reason specified above) 218·58, leaving the following errors:—

Date.	Error.
1865.	
June 8	+ 1·63
" "	− 0·02
" "	+ 1·76
" "	− 0·72
" "	+ 0·22
" 9	− 0·46
" "	− 0·86
" "	− 0·49
" "	− 1·39
" "	− 1·20
" 10	+ 1·60
" "	− 0·02

The sum of the squares of which is 13·48. Therefore the probable error of the mean is

$$\pm 0{\cdot}674\sqrt{\frac{13{\cdot}48}{132}} = \pm 0{\cdot}216;$$

consequently

$$I_B - I_b = 218{\cdot}58 \pm 0{\cdot}22 \tag{3}$$

The mean of the quantities in the last column of Table III. is 63·28, leaving the following system of errors :—

Date.	Error.	Date.	Error.
1865.		1865.	
June 14	− 1·42	June 17	+ 1·96
„ „	− 1·69	„ „	+ 0·06
„ 15	− 3·20	„ „	+ 1·54
„ „	− 1·25	„ „	− 1·08
„ „	− 2·62	„ „	+ 0·71
„ „	+ 0·26	„ 19	+ 4·19
„ 16	+ 1·12	„ „	− 1·64
„ „	+ 1·24	„ „	+ 0·72
„ „	− 0·65	„ „	− 0·98
„ „	+ 0·13	„ „	+ 2·57
„ 17	+ 0·03		

The sum of the squares of which is 63·06. Hence the probable error of the mean is

$$\pm 0{\cdot}674 \sqrt{\frac{63{\cdot}06}{420}} = \pm 0{\cdot}261\,;$$

consequently

$$\mathbf{I}_8 - \mathbf{O}_1 = 63{\cdot}28 \pm 0{\cdot}26 \tag{4}$$

Again; the mean of the quantities in the last column of Table IV. is 195·36, leaving the following system of errors :—

Date.	Error.	Date.	Error.
1865.		1865.	
June 12	+ 2·60	June 20	+ 2·18
„ „	+ 2·71	„ „	− 0·28
„ „	− 2·97	„ „	− 1·53
„ „	− 0·01	„ „	− 1·38
„ „	− 1·48	„ „	− 0·09
„ „	+ 0·89	„ „	+ 0·00
„ 13	+ 1·94	„ 21	− 1·64
„ „	+ 2·15	„ „	− 2·13
„ „	− 0·30	„ „	− 0·21
„ „	+ 1·31	„ „	− 1·84

The sum of the squares of which is 55·84. Therefore the probable error of the mean is

$$\pm 0{\cdot}674 \sqrt{\frac{55{\cdot}84}{380}} = \pm 0{\cdot}258\,;$$

consequently

$$\mathbf{I}_B - \mathbf{O}_1 = 195{\cdot}36 \pm 0{\cdot}26 \tag{5}$$

The probable errors of these results should be slightly increased on account of the probable errors of the adopted rates of expansion. The difference, however, is doubtless insensible, except in the case of $\mathbf{I}_b$, and even here the nearness of the mean of the temperatures to 62° will reduce the residual uncertainty to an insignificant amount.

4.

The four results — equations (2), (3), (4), (5)—at which we have arrived are not independent; that is to say, there is a relation amongst them which but for errors of observation should hold good. For instance, (4) − (2) and (5) − (3) give

$$\mathbf{I}_b - \mathbf{O}_1 = -23 \cdot 22 \ldots\ldots\ldots\ldots (4) - (2)$$
$$\mathbf{I}_b - \mathbf{O}_1 = -23 \cdot 22 \ldots\ldots\ldots\ldots (5) - (3)$$

and thus by a rare chance the four results are perfectly harmonious.

Again, it is equally remarkable that the relation of $\mathbf{I}_b$ and $\mathbf{O}_1$, here brought out, agrees all but precisely with the mean of the results of the comparisons between these bars in 1831 and 1846 as shown in equations (1).

This is all the more satisfactory when we remember that $\mathbf{I}_b$ has been subjected to a good deal of travelling, having been used in India during the interval 1831–1846, and subsequently sent from this country to Russia, where M. Struve compared it with several other geodetical standards.

XXI.

DETERMINATION OF THE LENGTH OF THE RUSSIAN DOUBLE TOISE

P.

1.

This bar is of wrought iron, an inch and a quarter square in section. It is an end-measure (*à bouts*); the terminal cylinders being of hard steel, and presenting at either end a small convex surface, a quarter of an inch in diameter for the contacts. The radius of curvature of these extreme surfaces is about 1·75 inch. The bar is contained in a wooden case 4·25 inches in breadth and 5 inches deep (external measurements), from which the ends of the bar project. The bar is supported at $\frac{1}{4}$ and $\frac{3}{4}$ of its length, not on rollers, but by passing through brass collars affixed to the box. One of the collars is round, the other square. In order to fit the circular collar, the bar is at that part *turned* in the form of a cylinder. This collar is in two pieces; the upper semicircle can be pressed down by means of a clamping screw, which projects through the box above, and so the bar is prevented from sliding longitudinally. The square collar fits the bar without play and without tightness. Two thermometers are let into the bar near the two points of support, they project upwards through the top of the box, their tubes being vertical. Each is protected by a small wooden case screwed to the bar box; this case has a glass front which is further covered when the thermometers are not being read by a (vertically) sliding cover. The whole length of the bar is carefully wrapped up in a cotton covering, and the space between the bar and its box is also entirely filled in with woollen material. Thus the bar is well protected from sudden changes of temperature.

In Plate X, figures 1, 2,

aaa . . . . is the box containing the bar.
dd . . . . are the projecting extremities of the bar.
ee . . . . are the covering boxes of the thermometers *tt*.
bb . . . . are handles by which the box is carried.
cc . . . . are iron plates fastened to the bottom of the box, and by which it rests on the supports.
σ . . . . is the clamping screw by means of which the bar is held firmly in the circular collar.

In order to support the box *aaa* properly during the comparisons of this bar with **OT**, a plank *ff* was constructed of mahogany, ten feet in length and eight inches in breadth. This plank rests immediately (figure 1) upon the carriages *gg*. Towards the extremities of the plank it is fitted with two (vertically) sliding brass frames *kk ll* seen in detail in figures 2, 3. *kk* are brass cylindrical rods connected by the cross pieces *ll'* above and below, forming a rigid rectangle; upon the upper cross piece, which is provided with flanges, the bar box rests. Motion in a vertical direction is communicated to this frame *kk ll'* by means of the steel rod *hh* which has a collar *i* working in a brass plate screwed to the upper surface of the mahogany plank. The thread of a screw which is cut on the upper end of *hh* works in the upper piece *l*, so that on turning the milled head *h* below the frame, *kk ll'* is moved up or down, all the weight being borne by the collar *i* of the steel rod.

In order to secure a vertical motion free from all shake, the steel rod is held below, near the milled head, by a strong bent plate of brass *mm* screwed to the lower surface of the plank. Thus it appears that when the bar-box is held upon these two sliding frames either end can be raised or lowered with a perfectly steady motion for either levelling or focusing.

Figure 2 shows one of the microscopes for reading the thermometers. Blocks *αα* (figure 1) are fastened to the mahogany plank; to these again plates of brass *ββ* (figures 1, 2) are screwed, but so as to admit of each in its own plane revolving round the screw which holds it. The microscopes *δ* are as described at the top of page 7; the brass slides on to which they are fitted, slide on to the pieces *ββ*. Thus by the movement of *β* round the screw which holds it, the microscope *δ* can be adjusted over the thermometer tube, and by means of its own brass slide the microscope can be brought over the end of the mercurial column so as to get it for observation precisely in the centre of the field.

In order to compare the Russian bar with **OT** it is necessary to use the contact apparatus. The mode of affixing it will be seen in figure 4. A strip of mahogany *nn*, four inches broad, is held fast to the box *aa* by means of the iron rings or bands *rr* (strips of iron bent into the form of a rectangle of 5 inches by 7 inches) which have pinching screws *ss* above; when these screws are loosed, the rings can be taken away. To the strip *nn* is fixed a block *n'* which has a brass plate *pp* screwed to its upper surface. This brass plate carries the contact apparatus C. It is held down to it by three vertical screws *uvv*, of which two only appear in the drawing, as two of the three, those two nearest the extremity *d* of the bar, are in line perpendicular to the bar. These last two screws pass freely through slotted holes in the plate *pp*, and screw into a loose strip of brass *q* below the plate; a groove being cut out of the mahogany *n'* to admit the piece of brass *q*. This piece being loose, when the two front screws *vv* are unclamped it will be seen that the contact piece admits of a movement in azimuth round the third screw *u*, the motion being communicated to it by means of the antagonistic screws *w*, of which only one appears in the figure.

2.

In order to compare the bar **P** with two lengths of **OT**, three microscopes were erected on the stone piers, H on the left pier, A on the centre pier, and K on the right pier; the micrometer head of A to the left. The bar **OT** (in its box) was then placed on the carriages and brought under H and A; it (the bar) was then most carefully levelled, and the microscopes adjusted to perfect focus with their zeros over the terminal lines of the toise, the axes of the microscopes being at the same time made perfectly vertical. The toise was then removed (by the running of the carriages along the rail) to under A and K. The microscope A remaining untouched, the bar is again in this position made truly level, while its left line is at the same time kept in focus of the microscope A. Then K is adjusted as to focus and verticality over the right line of the toise. Now supposing these operations accurately performed, the foci of the three microscopes will be found in one horizontal plane, and it remains to bring them into alignment or into a vertical plane. This is done by stretching a very fine thread of india-rubber to its full extent, and fixing its extremities so that the thread is bisected by the cross-hairs of the outer microscopes H and K. While the thread remains in this position the middle microscope is adjusted by the movement of its cast-iron stand until the cross-hairs bisect the thread.

These adjustments are then gone through a second time and perfected.

The height of the box of the Russian bar renders it necessary to remove the middle microscope A before that bar can be placed under H and K. This makes it needful that we be certain that if the microscope A be removed from its gun metal tube or holder (by releasing the springs behind) and returned again to it, it will take almost exactly the same position it had before. Repeated trials proved satisfactorily that this could be relied on within five divisions or so, which is abundantly accurate. For if the microscope, or rather the point bisected by its cross, should be out of place as much as a thousandth of an inch, or 25 divisions in a transverse direction, such displacement would not produce any sensible error in the comparisons; while a displacement in the direction of the bar's length is quite immaterial.

Another adjustment is that the transverse wires of the micrometers, perpendicular to the motion of the screw, must be perpendicular to the length of the toise, or parallel to the transverse lines on the platinum disks. Thus in the microscopes H and K we have the means of adjusting to the proper position in azimuth each of the contact pieces, so that the transverse lines shall be perpendicular to the length of the double toise.

It was assumed at first that the proper points of the terminal disks of the double toise at which the contacts should be made, were their centres; and several comparisons were made before this was found to be an error. For the intentional length of the bar is doubtless the distance between those tangent planes to the slightly convex disks which are perpendicular to the length of the bar,—or, the bar being horizontal, the vertical tangent planes which are also perpendicular to the bar's length. Now from the nature of the construction of the contact apparatus, as has been already explained, when the two parts are mounted together on their brass stand, the needles made precisely level, and the semi-cylinders at the same height, the contact of the convex surfaces is exactly half-way between the upper and lower horizontal surfaces of the semi-cylinders, and here the common tangent plane is by construction vertical. When the contact pieces are brought into contact with the disks of the toise, the needles being accurately levelled, and the contact made at the centre of the disks, the points of contact on the semi-cylinders are *above* the mid-depth, that is, are sensibly nearer the upper than the lower horizontal surface of the semi-cylinders, indicating that the tangent planes at the centres of the disks are not vertical, but converge (downwards), and that very nearly equally at the two extremities. In fact the proper point of contact on either of the disks—that is when the contact is precisely half-way between the upper and lower horizontal surfaces of the semi-cylinders—is above the centre of the disk, at the distance of about one tenth its diameter. As this determination is one of great importance it was considered necessary to have some separate and independent test, and to effect this a piece of sheet brass was cut out in the shape of a right-angled isosceles triangle; the edges containing the right angles were made perfectly straight, and the angle a perfect right angle, so far as could be ascertained by any mechanical tests. The brass plate *pp* (figure 4) was then made perfectly level by means of the small level of the contact apparatus, and one edge of the brass triangle being made to rest upon the horizontal plane, the other edge, truly vertical, was brought against the disk. The point of contact was precisely as indicated by the contact of the semi-cylinder. Further, on examining with a microscope the bright surface of the disks, there is, at one end unmistakeably, an appearance of *wear*, as though the contact had been habitually at a point above the centre, and dividing the vertical diameter in the proportion of about four to six.

There remains of course some little uncertainty as to the *precise* point where the contact should be made, but this within very narrow limits; and between these limits the contact was always kept, though frequently varied during the observations.

The observations have been made, in about equal proportions, by three observers. In each comparison two observers took part, the second observer B in each case also booking all the readings.

The manner of making a comparison will be best understood if the operations be tabulated thus :—

Order.	Observer.	Nature of Operation.	
1	B	Reads thermometers of **P**	**P**
2	A, B	Adjust **P** under H and K by the transverse screws of carriages and elevating or focus screws (*h* fig. 1)	
3	A	Reads H twice and K twice	
4	B	Reads the thermometers again	
5	A, B	Remove **P** from the carriages and substitute **O T**, adjusting it under H and A.	
6	B	Reads thermometers of **O T**	**O T**
7	A	Reads H twice and A twice	
8	A, B	Run **O T** under A and K and adjust for observation	
9	A	Reads A twice and K twice	
10	A	Reads A twice and K twice	**O T**
11	A, B	Run **O T** under H and A and adjust for observation	
12	A	Reads H twice and A twice	
13	B	Reads thermometers of **O T**	
14	A, B	Remove **O T** from carriages and substitute **P**, adjusting it under H and K	
15	A	Draws back needles of the contact apparatus from contact, corrects the levelling of needles, and renews the contact at each end ...	**P**
16	B	Reads thermometers of **P**	
17	A	Reads H twice and K twice	
18	B	Reads thermometers of **P**	

N.B.—**P** is left under the microscopes until next comparison.

No mention is made here of the removal and replacement of the microscope A. At the commencement of the operations, the bar **P** being under the microscopes, A is of course away.

Its replacement forms part of the operation 5, immediately on the removal of **P**, and its removal forms part of the operation 14, immediately before the removal of **O T** from the carriages.

The operations above specified occupy from 20 to 25 minutes. It will be seen that they involve two measurements of **P**, and two measurements of 2 **O T**.

The subjoined table shows the readings as recorded in one comparison or visit.

20th December 1865. 12h. 40m.

Thermⁿ. C.	P		O T		O T		Thermⁿ. F.
	H	K	H	A	A	K	
7°28	978·1	984·1	1115·2	26·7			44°83
7·16	978·3	984·1	1114·7	26·4			44·80
7·26					26·9	1124·0	
7·13					27·0	1123·5	
7·25					27·1	1123·0	
7·12					27·1	1123·1	
7·26	986·0	977·1	1116·4	28·0			44·85
7·13	986·9	977·3	1116·9	27·5			44·82

No. 39. Observers: Capt. Clarke, R.E., Corporal Compton, R.E.

We may here recount the different adjustments which require frequent renewal in order to exclude constant error.

1. The axes of revolution of the three microscopes to be vertical.
2. These axes to be in one vertical plane.
3. The outer foci of the microscopes to be in a horizontal plane.
4. Ordnance Toise to rest symmetrically on its rollers.
5. The steel needles of contact apparatus to be horizontal.
6. The point of contact to be at mid-depth of semi-cylinders.
7. The point of contact to be half-way between the parallel longitudinal lines of the contact apparatus, and so the tangent plane at the point of contact parallel to the transverse line on contact apparatus, or perpendicular to the bar's length.
8. Each piece of the contact apparatus to be level transversely.
9. No particle of dust to intercept contacts.
10. The bars to be adjusted to sharp focus.

These different adjustments were examined and renewed as often as practicable, so that from no one of them, nor any combination of them, can a constant error conceivably arise.

3.

In order to verify our received value of the interval between the lines on the contact apparatus, a series of measures was made precisely similar to that recorded at page 144. The results were as follows :—

SEPTEMBER 8TH, 1865.

Micrometer H.			Micrometer K.		
Left Line.	Right Line.	Diff.	Left Line.	Right Line.	Diff.
1350·30	639·63	710·67	646·33	1353·47	707·14
1350·67	639·73	710·94	643·70	1352·13	708·43
1350·53	639·43	711·10	644·43	1353·00	708·57
1349·33	637·97	711·36	642·10	1350·67	708·57
1354·90	644·00	710·90	641·00	1349·37	708·37

The mean of the measures by H is 710·99, which in millionths of a yard = 565·72. The mean of the measures by K is 708·22, which in millionths of a yard = 565·65. These may be considered very satisfactory checks upon the value, 565·62 of δ, equation 17), page 133.

4.

The thermometers of the Russian bar are divided to degrees centigrade. The scale is very small, 70° of the scale being exactly 3 inches in length, so that each degree is only $\frac{9}{70}$ of an inch in length. One degree Fahrenheit on this scale would only $= \frac{3}{70} \cdot \frac{5}{9} = \frac{1}{42}$ inch, whilst in the thermometers used in our own bars one degree on the scale is about $\frac{1}{5}$ of an inch—more than eight times as large. This renders it necessary that the errors of the thermometers should be very accurately obtained, and that they should be read very often in the comparisons. For, according to the determination of M. Struve, at page 49 of the first volume of his account of the Russian Arc of Meridian, the absolute expansion of **P** for one degree centigrade is 11·253 ± ·017 millionths of its length, or which is the same thing 26·651 millionths of a yard for each degree Fahrenheit. Consequently, an error in the reading of the thermometers of $\frac{1}{200}$ of an inch on the scale would be equivalent to an error in the length of the bar of $\frac{42}{200}$ 26·65 = 5·60. This serves to show how much depends on the accuracy of the readings of the thermometers and the determination of their errors.

These thermometers, which are numbered 7 and 12, were compared with the standards 3241 and 4142 both before and after the comparisons at the vicinity of the normal temperature, and in order to show the degree of accuracy attainable in reading the thermometers we shall here give the results of the comparisons. The Russian thermometers were suspended in the water trough, their tubes being accurately vertical as proved by a plumb line, and their bulbs at mid-depth of the water, while the standards as usual were in a horizontal position. In order to allow of the Russian thermometers being properly read, a rectangular aperture was cut in the side of the water trough, and this filled with a sheet of plate glass. A microscope was then mounted in a horizontal position for reading these thermometers: the manner of mounting the microscope was the same as in Fig. 2, Pl. X., described already. In order to render the observing through this microscope possible, the water trough was mounted upon trestles 14 inches high. The standards were read as usual with the long vertical microscope. It was found that the thermometers 7 and 12 could be read with tolerable certainty, and that an error of half a tenth of a degree in a single reading was scarcely probable.

We now give the results of the observations:—

Date.	Russian.		Standards.		Corrections to		Corrected Readings of		True Temperature.		Errors of	
	7	12	3241	4142	3241	4142	3241	4142	F	C	7	12
1865.	°	°	°	°			°	°	°	°		
July 17	15·049	14·968	59·013	58·493	−·59	−·09	58·423	58·403	58·413	14·674	+·375	+·294
,, ,,	17·196	17·073	63·004	62·511	−·60	−·11	62·404	62·401	62·403	16·891	+·305	+·182
,, ,,	17·324	17·139	63·108	62·601	−·60	−·11	62·508	62·491	62·500	16·944	+·380	+·195
,, ,,	17·951	17·845	64·265	63·795	−·61	−·11	63·655	63·685	63·670	17·594	+·357	+·251
,, ,,	18·704	18·571	65·638	65·136	−·61	−·11	65·028	65·026	65·027	18·348	+·356	+·223
Oct. 3	15·216	15·113	59·369	58·853	−·59	−·09	58·779	58·763	58·771	14·873	+·343	+·240
,, ,,	16·051	15·983	60·971	60·453	−·59	−·10	60·381	60·353	60·367	15·759	+·292	+·224
,, ,,	16·658	16·510	61·855	61·370	−·60	−·10	61·255	61·270	61·262	16·257	+·401	+·253
,, ,,	17·093	17·010	62·748	62·260	−·60	−·11	62·148	62·150	62·149	16·749	+·344	+·261
,, ,,	17·674	17·528	63·723	63·225	−·61	−·11	63·113	63·115	63·114	17·286	+·388	+·242
,, ,,	18·108	18·023	64·578	64·095	−·61	−·11	63·968	63·985	63·976	17·764	+·344	+·259

The second, third, fourth, and fifth columns are the means of the actual readings of the four thermometers; each line is the mean of four comparisons, with the exception of the first two lines, which are each the mean of five comparisons.

Suppose, now, as there is no sufficient evidence to the contrary, that the errors of thermometers 7 and 12 are constant within the range of temperature exhibited in these observations, and that the discrepancies in the last two columns are due to the unavoidable errors of observation; then the error of thermometer 7 is—in degrees centigrade—

$$+0^\circ{\cdot}353 \pm {\cdot}0067$$

and the error of 12 is

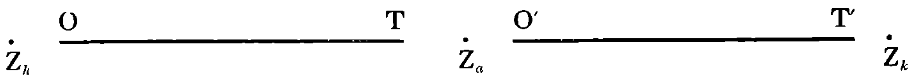
$$+0^\circ{\cdot}239 \pm {\cdot}0064$$

while the probable error of a single determination in either thermometer (represented by the mean of 8 readings) is about $\pm 0{\cdot}22$. The mean of the two thermometers should then in the long run give the true temperature with a probable error of $\pm{\cdot}0046$ of a degree centigrade. This corresponds to a probable error of $\pm 0{\cdot}22$ of a millionth of a yard in the length of the bar. To this should be added the probable error of the mean of all the absolute temperatures given by the two Standard thermometers in the above observations; but this, being a very much smaller quantity, we neglect.

The errors of thermometers 7 and 12 were also obtained immediately previous to the comparisons at 44°.

5.

We shall now consider the reduction of the observations. In the accompanying diagram



$Z_h O = h$ $\quad$ $T Z_a = -a_1$: $Z_a O' = a_2$ $\quad$ $T' Z_k = k$

let the dots Z_h Z_a Z_k be the zeros of the microscopes H A K; OT O′T′ the extremities of **OT** in its two positions. In the first position h and a_1 are the readings (reduced to the proper unit) of H and A; in the second position a_2 and k are the readings, reduced, of A and K. Referring to the table, page 259, h and k are the means of the fourth and seventh columns $-$ 1000 (the reading of centre of field) reduced, a_1 a_2 are the means, reduced, of the fifth and sixth columns. Let also h' k' be the means $-$ 1000, reduced, of the second and third columns. Then we have

$$Z_h Z_a = \mathbf{OT} + h - a_1$$
$$Z_a Z_k = \mathbf{OT} + k + a_2$$
$$\therefore Z_h Z_k = 2\,\mathbf{OT} + h + k + a_2 - a_1 \qquad (1)$$

Again, when the bar **P** is under the microscopes, if h' k' be the reduced readings of H and K, then σ being the value of the contact space

$$Z_h Z_k = \mathbf{P} + \sigma + h' + k' \qquad (2)$$

Therefore taking the difference of (1) and (2) and putting $\mathbf{T}_o$ for the length of the Ordnance Toise at the observed temperature

$$\mathbf{P} + \sigma - 2\,\mathbf{T}_o = h - h' + k - k' + a_2 - a_1 \qquad (3)$$

which we shall write thus, since $\sigma = 565 \cdot 85$,

$$\mathbf{P} - 2\,\mathbf{T}_o = n - 565 \cdot 85 \qquad (4)$$

Let the length of the Russian bar be expressed generally by the equation

$$\mathbf{P} = \alpha + \beta\,(t - 62)$$

and the length of the Ordnance Toise by the equation

$$\mathbf{T}_o = \alpha_o + \beta_o\,(t_o - 62)$$

then

$$\mathbf{P} - 2\,\mathbf{T}_o = \alpha - 2\,\alpha_o + \beta\,(t - 62) - 2\,\beta_o\,(t_o - 62) \qquad (5)$$

where t is the temperature, reduced to Fahrenheit, of the Russian bar, and t_o the temperature of the Ordnance Toise.

For convenience in our subsequent reductions put

$$\alpha - 2\,\alpha_o = x - 312 \cdot 35 \qquad (6)$$

$$\beta - 2\,\beta_o = y \qquad (7)$$

Substituting these in (5)

$$\mathbf{P} - 2\,\mathbf{T}_o = x + y\,(t_o - 62) + \beta\,(t - t_o) - 312 \cdot 35 \qquad (8)$$

Subtracting (4) from (8) we get

$$x + y\,(t_o - 62) + 253 \cdot 50 + \beta\,(t - t_o) - n = 0 \qquad (9)$$

Or putting,

$$t_o - 62 = f \qquad (10)$$

$$253 \cdot 50 + \beta\,(t - t_o) - n = k$$

we have finally

$$x + fy + k = 0 \qquad (11)$$

Each comparison will supply an equation of this form, and these equations being treated according to the method of least squares, the values x and y are determined, and thence the difference $(\alpha - 2\,\alpha_o)$ of the length $\mathbf{P} - 2\mathbf{T}_o$ when both bars are at 62°.

6.

The results of the different comparisons are shown in the first of the following tables which gives the means of the micrometer readings and the corrected means of the thermometer readings.

The second table contains the different steps of the reduction (which will be understood from the headings of the different columns) up to the values of f and k in equations (10).

I.

Date.	P			OT		
	Micrometer Readings.	Temp.	Equiv.	Micrometer Readings.	Micrometer Readings.	Temp.
1865.		C °	F °			F °
Sept. 12	104·73*h* + 99·85*k*	17·946	64·30	255·70*h* — 38·78*a*	39·18*a* + 260·28*k*	64·22
„ „	94·45*h* + 104·48*k*	18·039	64·47	248·55*h* — 30·40*a*	30·90*a* + 263·30*k*	64·42
„ „	100·85*h* + 96·10*k*	18·173	64·71	255·33*h* — 34·73*a*	35·48*a* + 253·80*k*	64·62
„ 13	102·93*h* + 99·08*k*	18·137	64·65	257·13*h* — 26·48*a*	27·38*a* + 257·58*k*	64·47
„ „	94·60*h* + 104·38*k*	18·172	64·71	254·50*h* — 6·40*a*	7·60*a* + 256·00*k*	64·66
„ „	98·90*h* + 93·88*k*	18·258	64·86	252·48*h* — 30·10*a*	30·70*a* + 252·30*k*	64·82
„ 14	106·40*h* + 90·88*k*	18·164	64·70	256·43*h* — 41·58*a*	42·53*a* + 256·70*k*	64·60
„ „	99·23*h* + 98·25*k*	18·219	64·79	257·23*h* — 25·53*a*	26·93*a* + 252·18*k*	64·74
„ „	96·10*h* + 99·38*k*	18·283	64·91	250·95*h* — 28·10*a*	28·18*a* + 256·58*k*	64·81
„ 15	99·75*h* + 117·03*k*	17·816	64·07	259·88*h* — 23·90*a*	27·13*a* + 270·00*k*	63·93
„ „	107·25*h* + 108·33*k*	17·861	64·15	262·43*h* — 29·53*a*	29·43*a* + 266·95*k*	64·15
„ „	99·75*h* + 106·53*k*	17·996	64·39	259·70*h* — 38·43*a*	39·53*a* + 261·13*k*	64·39
„ 23	114·43*h* + 107·85*k*	16·813	62·26	274·10*h* — 41·08*a*	43·58*a* + 257·50*k*	62·13
„ „	110·90*h* + 110·95*k*	16·799	62·24	266·53*h* — 36·30*a*	37·40*a* + 266·10*k*	62·18
„ „	104·08*h* + 119·00*k*	16·808	62·25	267·20*h* — 40·80*a*	41·48*a* + 265·68*k*	62·17
„ 25	128·93*h* + 128·73*k*	16·018	60·82	285·43*h* — 46·43*a*	47·05*a* + 275·78*k*	60·81
„ „	125·43*h* + 129·65*k*	16·061	60·91	282·13*h* — 39·43*a*	39·53*a* + 278·65*k*	60·96
„ „	127·98*h* + 122·28*k*	16·173	61·11	273·88*h* — 32·03*a*	33·28*a* + 280·73*k*	61·11
„ 26	124·35*h* + 132·30*k*	15·958	60·72	283·15*h* — 52·75*a*	53·78*a* + 279·70*k*	60·69
„ „	116·68*h* + 138·33*k*	16·023	60·84	282·23*h* — 26·58*a*	26·15*a* + 279·50*k*	60·83
„ „	120·65*h* + 134·45*k*	16·004	60·81	270·60*h* — 30·55*a*	30·28*a* + 291·48*k*	60·85
„ 27	126·83*h* + 133·78*k*	15·819	60·47	278·50*h* — 2·48*a*	2·65*a* + 288·73*k*	60·49
„ „	132·20*h* + 130·43*k*	15·913	60·64	290·43*h* — 38·28*a*	39·28*a* + 276·78*k*	60·68
„ „	125·48*h* + 133·68*k*	16·007	60·81	274·65*h* — 33·00*a*	33·43*a* + 289·55*k*	60·80
„ 28	125·13*h* + 143·08*k*	15·776	60·40	287·10*h* — 33·98*a*	34·50*a* + 284·38*k*	60·37
„ „	127·93*h* + 143·83*k*	15·834	60·50	288·43*h* — 33·20*a*	32·65*a* + 290·53*k*	60·52
„ „	128·25*h* + 141·83*k*	15·831	60·50	284·85*h* — 41·48*a*	41·43*a* + 289·08*k*	60·52
„ 29	135·15*h* + 144·70*k*	15·661	60·19	300·00*h* — 47·40*a*	47·48*a* + 286·83*k*	60·08
„ „	139·60*h* + 139·73*k*	15·698	60·26	301·10*h* — 46·93*a*	48·10*a* + 281·55*k*	60·19
„ „	140·10*h* + 138·43*k*	15·709	60·28	290·05*h* — 46·30*a*	45·85*a* + 293·95*k*	60·25
Dec. 16	75·25*h* + 87·65*k*	6·189	43·14	226·50*h* — 14·33*a*	14·20*a* + 215·83*h*	42·94
„ 18	63·13*h* + 84·93*k*	6·538	43·77	215·03*h* — 13·08*a*	14·80*a* + 204·50*h*	43·72
„ „	64·95*h* + 76·68*k*	6·641	43·95	209·40*h* — 7·00*a*	8·20*a* + 203·40*h*	43·99
„ „	63·68*h* + 71·53*k*	6·721	44·10	205·30*h* — 15·98*a*	15·85*a* + 205·45*h*	44·19
„ 19	82·78*h* + 86·30*k*	6·739	44·13	218·23*h* — 15·05*a*	15·60*a* + 226·70*k*	44·19
„ „	80·85*h* + 82·45*k*	6·816	44·27	213·60*h* — 29·93*a*	30·38*a* + 222·18*k*	44·40
„ „	80·60*h* + 80·95*k*	6·920	44·46	218·25*h* — 29·48*a*	29·98*a* + 214·95*k*	44·58
„ 20	78·65*h* + 88·20*k*	6·835	44·30	213·95*h* — 19·65*a*	20·13*a* + 230·65*k*	44·36
„ „	82·33*h* + 80·65*k*	6·919	44·45	215·80*h* — 27·15*a*	27·03*a* + 223·40*k*	44·54
„ „	82·23*h* + 77·90*k*	6·995	44·59	214·20*h* — 41·63*a*	41·90*a* + 218·88*h*	44·73

II.

Date.		Difference of Length in Micrometer Divisions.	n	$253·50 - n$	$t-t_0$	$(t-t_0)\delta$	f	k
1865.								
Sept.	12	150·97 *h* + 0·40 *a* + 160·43 *k*	248·73	+ 4·77	+0·08	+2·13	+ 2·22	+ 6·90
”	”	154·10 *h* + 0·50 *a* + 158·82 *k*	250·05	3·45	+0·05	+1·33	+ 2·42	4·78
”	”	154·48 *h* + 0·75 *a* + 157·70 *k*	249·75	3·75	+0·09	+2·40	+ 2·62	6·15
”	13	154·20 *h* + 0·90 *a* + 158·50 *k*	250·34	3·16	+0·18	+4·80	+ 2·47	7·96
”	”	159·90 *h* + 1·20 *a* + 151·62 *k*	249·74	3·76	+0·05	+1·33	+ 2·66	5·09
”	”	153·58 *h* + 0·60 *a* + 158·42 *k*	249·43	4·07	+0·04	+1·07	+ 2·82	5·14
”	14	150·03 *h* + 0·95 *a* + 165·82 *k*	252·93	0·57	+0·10	+2·67	+ 2·60	3·24
”	”	158·00 *h* + 1·40 *a* + 153·93 *k*	250·31	3·19	+0·05	+1·33	+ 2·74	4·52
”	”	154·85 *h* + 0·08 *a* + 157·20 *k*	248·86	4·64	+0·10	+2·67	+ 2·81	7·31
”	15	160·13 *h* + 3·23 *a* + 152·97 *k*	253·39	0·11	+0·14	+3·73	+ 1·93	3·84
”	”	155·18 *h* − 0·10 *a* + 158·62 *k*	250·04	3·46	0·00	0·00	+ 2·15	3·46
”	”	159·95 *h* + 1·10 *a* + 154·60 *k*	252·04	1·46	0·00	0·00	+ 2·39	1·46
”	23	159·67 *h* + 2·50 *a* + 149·65 *k*	249·51	3·99	+0·13	+3·46	+ 0·13	7·45
”	”	155·63 *h* + 1·10 *a* + 155·15 *k*	249·04	4·46	+0·06	+1·60	+ 0·18	6·06
”	”	163·12 *h* + 0·68 *a* + 146·68 *k*	247·74	5·76	+0·08	+2·13	+ 0·17	7·89
”	25	156·50 *h* + 0·62 *a* + 147·05 *k*	242·70	10·80	+0·01	+0·27	− 1·19	11·07
”	”	156·70 *h* + 0·10 *a* + 149·00 *k*	243·81	9·69	−0·05	−1·33	− 1·04	8·36
”	”	145·90 *h* + 1·25 *a* + 158·45 *k*	244·11	9·39	0·00	0·00	− 0·89	9·39
”	26	158·80 *h* + 1·03 *a* + 147·40 *k*	245·29	8·21	+0·03	+0·80	− 1·31	9·01
”	”	165·55 *h* − 0·43 *a* + 141·17 *k*	243·97	9·53	+0·01	+0·27	− 1·17	9·80
”	”	149·95 *h* − 0·27 *a* + 157·03 *k*	244·41	9·09	−0·04	−1·07	− 1·15	8·02
”	27	151·67 *h* + 0·17 *a* + 154·95 *k*	244·64	8·86	−0·02	−0·53	− 1·51	8·33
”	”	158·23 *h* + 1·00 *a* + 146·35 *k*	243·97	9·53	−0·04	−1·07	− 1·32	8·46
”	”	149·17 *h* + 0·43 *a* + 155·87 *k*	243·69	9·81	+0·01	+0·27	− 1·20	10·08
”	28	161·97 *h* + 0·52 *a* + 141·30 *k*	242·34	11·16	+0·03	+0·80	− 1·63	11·96
”	”	160·50 *h* − 0·55 *a* + 146·70 *k*	244·23	9·27	−0·02	−0·53	− 1·48	8·74
”	”	156·60 *h* − 0·05 *a* + 147·25 *k*	242·15	11·35	−0·02	−0·53	− 1·48	10·82
”	29	164·85 *h* + 0·08 *a* + 142·13 *k*	244·78	8·72	+0·11	+2·93	− 1·92	11·65
”	”	161·50 *h* + 1·17 *a* + 141·82 *k*	243·15	10·35	+0·07	+1·87	− 1·81	12·22
”	”	149·95 *h* − 0·45 *a* + 155·52 *k*	242·99	10·51	+0·03	+0·80	− 1·75	11·31
Dec.	16	151·25 *h* − 0·13 *a* + 128·18 *k*	222·57	30·93	+0·20	+5·33	−19·06	36·26
”	18	151·90 *h* + 1·72 *a* + 119·57 *k*	218·39	35·11	+0·05	+1·33	−18·28	36·44
”	”	144·45 *h* + 1·20 *a* + 126·72 *k*	217·56	35·94	−0·04	−1·07	−18·01	34·87
”	”	141·62 *h* − 0·13 *a* + 133·92 *k*	219·49	34·01	−0·09	−2·40	−17·81	31·61
”	19	135·45 *h* + 0·55 *a* + 140·40 *k*	220·56	32·94	−0·06	−1·60	−17·81	31·34
”	”	132·75 *h* + 0·45 *a* + 139·73 *k*	217·76	35·74	−0·13	−3·46	−17·60	32·28
”	”	137·65 *h* + 0·50 *a* + 134·00 *k*	217·14	36·36	−0·12	−3·20	−17·42	33·16
”	20	135·30 *h* + 0·48 *a* + 142·45 *k*	221·99	31·51	−0·06	−1·60	−17·64	29·91
”	”	133·47 *h* − 0·12 *a* + 142·75 *k*	220·07	33·43	−0·09	−2·40	−17·46	31·03
”	”	131·97 *h* + 0·27 *a* + 140·98 *k*	217·92	+35·58	−0·14	−3·73	−17·27	+31·85

It is unnecessary to write down the equations of condition as they are obtained at a glance from the last two columns of the second table. The final equations in x and y are—

$$40\,x - 168\cdot900\,y + 559\cdot22 = 0 \qquad (12)$$
$$-168\cdot90\,x + 3288\cdot987\,y - 5929\cdot10 = 0$$

or putting A and B for the absolute terms—

$$x + \cdot03192190\,A + \cdot001639291\,B = 0 \qquad (13)$$
$$y + \cdot00163929\,A + \cdot000388228\,B = 0$$

whence

$$x = -\,8\cdot13 \;\ldots\ldots\; \textit{Reciprocal of weight} = \cdot031922 \qquad (14)$$
$$y = +\,1\cdot385 \qquad ,, \quad ,, \quad = \cdot000388$$

Substituting these values in the equations of condition we get the following system of errors :—

Date.		Error.	Date.		Error.	Date.		Error.	Date.		Error.
1865.			1865.			1865.			1865:		
Sept.	12	+ 1·84	Sept.	15	− 1·69	Sept.	26	− 1·70	Dec.	16	+ 1·73
,,	,,	0·00	,,	,,	− 3·36	,,	27	− 1·89	,,	18	+ 2·99
,,	,,	+ 1·65	,,	23	− 0·50	,,	,,	− 1·50	,,	,,	+ 1·79
,,	13	+ 3·25	,,	,,	− 1·82	,,	,,	+ 0·29	,,	,,	− 1·19
,,	,,	+ 0·64	,,	,,	+ 0·01	,,	28	+ 1·57	,,	19	− 1·46
,,	,,	+ 0·92	,,	25	+ 1·29	,,	,,	− 1·44	,,	,,	− 0·23
,,	14	− 1·29	,,	,,	− 1·21	,,	,,	+ 0·64	,,	,,	+ 0·90
,,	,,	+ 0·19	,,	,,	+ 0·04	,,	29	+ 0·86	,,	20	− 2·65
,,	,,	+ 3·07	,,	26	− 0·93	,,	,,	+ 1·58	,,	,,	− 1·28
,,	15	− 1·62	,,	,,	+ 0·05	,,	,,	+ 0·76	,,	,,	− 0·20

The sum of the squares of these errors is $99\cdot19$, consequently the probable error of a single comparison is

$$\pm\,0\cdot674\sqrt{\frac{99\cdot19}{40-2}} = \pm\,1\cdot089 \qquad (15)$$

and the probable errors of x and y

$$x \;\ldots\ldots\ldots\; \pm\,1\cdot089\,\sqrt{\cdot03192} = \pm\,0\cdot195 \qquad (16)$$
$$y \;\ldots\ldots\ldots\; \pm\,1\cdot089\,\sqrt{\cdot000388} = \pm\,0\cdot021$$

combining equations (6) and (14) we have for the relative lengths of **P** and $\mathbf{T}_0$ when they are both at the temperature of 62°

$$\mathbf{P} - 2\,\mathbf{T}_0 = \alpha - 2\alpha_0 = -\,320\cdot48 \qquad (17)$$

But taking into account the probable error of x the precise result we have obtained is

$$\mathbf{P} = 2\,\mathbf{T}_0 - \sigma + 245\cdot37 \pm 0\cdot195 \qquad (18)$$

We have seen that the probable errors of the determinations of the corrections to be applied to the Russian thermometers correspond to a probable error in the length of the bar of $\pm\,0\cdot22$. Also the probable error of σ is $\pm\cdot108$; hence

$$\mathbf{P} - 2\,\mathbf{T}_0 = -\,320\cdot48 \pm \sqrt{(\cdot22)^2 + (\cdot108)^2 + (\cdot195)^2} \qquad (19)$$

or

$$\mathbf{P} = 2\,\mathbf{T}_0 - 320\cdot48 \pm 0\cdot313 \qquad (20)$$

7.

We may now express **P** in terms of $\mathbf{Y}_{55}$, for from equation (16), page 144,

$$2\,\mathbf{T}_0 = (4 \cdot 26335024 \pm \cdot 00000043)\ \mathbf{Y}_{55}$$

whence

$$\mathbf{P} = 4 \cdot 26302976\ \mathbf{Y}_{55}$$

To get the probable error of this result we must remember that the probable errors of $2\,\mathbf{T}_0$ and of σ are connected and not independent, and we must proceed as at pages 134, 135.

The expression for $2\,\mathbf{T}_0 - \sigma$ is found to be

$$\frac{15346}{3600}\mathbf{Y}_{55} + \frac{473}{600}\lambda_1 + 2\,\lambda_2 + 2\,\lambda_3 - \lambda_4 + \frac{474}{600}\,b\left(\epsilon_1 + \frac{\epsilon^2}{3}\right) - \frac{4}{5}\,x_2 - 2\,x_\tau - \frac{6}{5}x_3$$
$$+ \frac{1}{5}\,x_5 - \frac{1}{5}\,x_7 - \frac{27}{50}\,x_b + 2\,x_f + \frac{473}{300}\,x_g$$

The following table contains the arrangement of this formula in its independent parts and the corresponding probable errors:

No. of independent Operations.	$2\,\mathbf{T}_0 - \sigma$	Partial Probable Errors.
1	$\frac{473}{300}\,x_g$	$\pm \cdot 058$
2	$- \frac{27}{50}\,x_b + 2\,x_f$	$\pm \cdot 074$
3	$- \frac{4}{5}\,x_2 - \frac{6}{5}\,x_3 + \frac{1}{5}\,x_6 - \frac{1}{5}\,x_7$	$\pm \cdot 094$
4	$- 2\,x_\tau$	$\pm \cdot 196$
5	$\frac{473}{600}\,\lambda_1 + \frac{474}{600}\,b\,\epsilon_1$	$\pm \cdot 088$
6	$2\,\lambda_2 + \frac{158}{600}\,b\,\epsilon_2$	$\pm \cdot 368$
7	$2\,\lambda_3$	$\pm \cdot 128$
8	$-\ \lambda_4$	$\pm \cdot 107$

To these it remains to add the probable error of x, equation (16), and that resulting from the comparisons of the Russian thermometers. Thus the square of the probable error of **P** in terms of $\mathbf{Y}_{55}$ is

$$(058)^2 + (074)^2 + (094)^2 + (196)^2 + (088)^2 + (368)^2 + (128)^2 + (107)^2 + (195)^2$$
$$+ (220)^2 = (560)^2$$

Therefore, finally, **P** and $\mathbf{Y}_{55}$ being both at the temperature of 62° Fahrenheit

$$\mathbf{P} = (4 \cdot 26302976 \pm \cdot 00000056)\ \mathbf{Y}_{55} \qquad (21)$$

XXII.

FINAL RESULTS.

1.

TEN FEET STANDARD BARS.

Between the comparisons made amongst the five bars $\mathbf{O}_1$, $\mathbf{OI}_1$, $\mathbf{I}_S$, $\mathbf{I}_B$, $\mathbf{I}_b$, there exist relations such as make their absolute lengths mutually dependent, for $\mathbf{OI}_1$ and $\mathbf{I}_S$ have both been compared with $\mathbf{Y}_{55}$; $\mathbf{O}_1$ has been compared with $\mathbf{OI}_1$, with $\mathbf{I}_S$, and with $\mathbf{I}_B$; $\mathbf{I}_b$ has been compared with $\mathbf{I}_S$ and $\mathbf{I}_B$. The differences are as follow :—

$$\begin{aligned}
\mathbf{OI}_1 - \tfrac{10}{3}\,\mathbf{Y}_{55} &= 21\cdot08 \quad \ldots\ldots \text{ page } 92 \\
\mathbf{I}_S - \tfrac{10}{3}\,\mathbf{Y}_{55} &= 70\cdot62 \quad \ldots\ldots \quad ,, \quad 241 \\
\mathbf{OI}_1 - \mathbf{O}_1 &= 18\cdot38 \quad \ldots\ldots \quad ,, \quad 93 \\
\mathbf{I}_S - \mathbf{O}_1 &= 63\cdot28 \quad \ldots\ldots \quad ,, \quad 254 \\
\mathbf{I}_B - \mathbf{O}_1 &= 195\cdot36 \quad \ldots\ldots \quad ,, \quad 254 \\
\mathbf{I}_S - \mathbf{I}_b &= 86\cdot50 \quad \ldots\ldots \quad ,, \quad 253 \\
\mathbf{I}_B - \mathbf{I}_b &= 218\cdot58 \quad \ldots\ldots \quad ,, \quad 253
\end{aligned} \qquad (1)$$

Now, let the most probable lengths of the bars be,—

$$\begin{aligned}
\mathbf{OI}_1 &= \tfrac{10}{3}\,\mathbf{Y} + x_1 \\
\mathbf{I}_S &= \tfrac{10}{3}\,\mathbf{Y} + x_2 \\
\mathbf{O}_1 &= \tfrac{10}{3}\,\mathbf{Y} + x_3 \\
\mathbf{I}_B &= \tfrac{10}{3}\,\mathbf{Y} + x_4 \\
\mathbf{I}_b &= \tfrac{10}{3}\,\mathbf{Y} + x_5
\end{aligned} \qquad (2)$$

then these values substituted in equation (1) give the following—

$$\begin{aligned}
x_1 - 21\cdot08 &= 0 \\
x_2 - 70\cdot62 &= 0 \\
x_1 - x_3 - 18\cdot38 &= 0 \\
x_2 - x_3 - 63\cdot28 &= 0 \\
x_4 - x_3 - 195\cdot36 &= 0 \\
x_2 - x_5 - 86\cdot50 &= 0 \\
x_4 - x_5 - 218\cdot58 &= 0
\end{aligned} \qquad (3)$$

Solving by the method of least squares, we get—

$$
\begin{array}{rrrrrrl}
2x_1 & & -x_3 & & & -39\cdot46 = 0 & \quad (4)\\
& 3x_2 & -x_3 & & -x_5 & -220\cdot40 = 0 & \\
-x_1 & -x_2 & +3x_3 & -x_4 & & +277\cdot02 = 0 & \\
& & -x_3 & +2x_4 & -x_5 & -413\cdot94 = 0 & \\
& -x_2 & & -x_4 & +2x_5 & +305\cdot08 = 0 &
\end{array}
$$

The values of x_1, x_2, x_3, x_4, x_5, which satisfy these equations, are—

$$
\begin{aligned}
x_1 &= +\ 22\cdot32 \qquad (5)\\
x_2 &= +\ 69\cdot38\\
x_3 &= +\ 5\cdot17\\
x_4 &= +\ 200\cdot84\\
x_5 &= +\ 17\cdot43
\end{aligned}
$$

And the residual errors of equations (3) are—

$$
\begin{array}{ccc}
+1\cdot24 & -1\cdot23 & +0\cdot31 \\
-1\cdot24 & +0.93 & -0\cdot31 \\
 & +0\cdot31 &
\end{array} \qquad (6)
$$

These errors are greater than we should have been led to expect from the probable errors of the seven quantities exhibited in equations (1) (for these probable errors see the pages referred to). The inference is, that in some, or perhaps all, of the different series of comparisons—seven in number—expressed in these equations, an unknown source of error, constant or inconstant, has existed.

The sum of the squares of the errors (6) is $5\cdot7413$; hence the probable error of one equation is—

$$\pm 0\cdot674\sqrt{\frac{5\cdot7413}{7-5}} = \pm 1\cdot14$$

and for $x_1 \ldots\ldots x_5$ the resulting probable errors are—

$$
\begin{aligned}
x_1 \text{ or } x_2 \ldots\ldots\ldots & \pm 1\cdot14\sqrt{\frac{11}{15}} = \pm 0\cdot98 \qquad (7)\\
x_3 \ldots\ldots\ldots & \pm 1\cdot14\sqrt{\frac{14}{15}} = \pm 1\cdot10\\
x_4 \ldots\ldots\ldots & \pm 1\cdot14\sqrt{\frac{21}{15}} = \pm 1\cdot35\\
x_5 \ldots\ldots\ldots & \pm 1\cdot14\sqrt{\frac{20}{15}} = \pm 1\cdot32
\end{aligned}
$$

But the number of supernumerary equations here is altogether too small to give reliable probable errors to the results. We therefore only adopt these last numbers (7) as *approximate*, and with this reserve state the final lengths as follows :—

Ordnance Intermediate Standard ... $\mathbf{OI}_1 = (3 \cdot 33335565 \pm \cdot 00000098)\ \mathbf{Y}_{55}$
Ordnance Survey Standard........... $\mathbf{O}_1 = (3 \cdot 33333850 \pm \cdot 00000110)\ \mathbf{Y}_{55}$
Indian (Steel) Standard .... $\mathbf{I}_S = (3 \cdot 33340271 \pm \cdot 00000098)\ \mathbf{Y}_{55}$
Indian (Bronze) Standard........... $\mathbf{I}_B = (3 \cdot 33353417 \pm \cdot 00000135)\ \mathbf{Y}_{55}$
Old Indian Standard B.............. $\mathbf{I}_b = (3 \cdot 33331590 \pm \cdot 00000132)\ \mathbf{Y}_{55}$

2.

THE TOISE.

The unit of length in which the greater part of the European Geodetical Measurements are expressed is the Toise, the actual standard being the bar known as the *Toise of Peru.* This standard was constructed in 1735 for the measurement of the Arc of Peru by MM. Bouguer and de la Condamine.* It is a flat bar of polished iron, 1·51 inch in width, and 0·4 inch in thickness, notched at the ends, as in the figure below, which represents the Toise in

plan (curtailed). The length is measured from the face *a c* to the face *b e*. The projections *a d*, *b f*, serve to protect the surfaces *a c*, *b e*, from accidents. The temperature at which this bar represents *the Toise* is—

$$13^\circ \cdot 00\ Reaum^r = 16^\circ \cdot 25\ Cent^e = 61^\circ \cdot 25\ Fahr^t.$$

The coefficient of its expansion has never been determined by direct observation.

---

* "Nous avions emporté avec nous, en 1735, une règle de fer poli, de dix-sept lignes de largeur sur quatre " lignes et demie d'épaisseur. M. *Godin*, aidé d'un artiste habile, avoit mis toute son attention à ajuster " la longueur de cette règle sur celle de la Toise *étalon*, qui a été fixée en 1668 au pied de l'escalier du grand " Châtelet de Paris. Je prévis que cet ancien étalon, fait assez grossièrement, et d'ailleurs exposé aux " chocs, aux injures de l'air, à la rouille, au contact de toutes les mesures qui y sont présentées, et à la " malignité de tout mal-intentionné, ne seroit guère propre à vérifier dans la suite la Toise qui alloit servir à la " mesure de la Terre, et devenir l'original auquel les autres devoient être comparées. Il me parut donc " très-nécessaire en emportant une Toise bien vérifiée, d'en laisser à Paris une autre de même matière et de " même forme, à laquelle on pût avoir recours s'il arrivoit quelqu'accident à la nôtre pendant un si long voyage. " Je me chargeai d'office du soin d'en faire faire une toute pareille. Cette seconde Toise fut construite par le " même ouvrier, et avec les mêmes précautions que la première. Les deux Toises furent comparées ensemble " dans une de nos assemblées, et l'une des deux resta en dépôt à l'Académie; c'est la même qui a été depuis " portée en Lapponie par M. de *Maupertuis*, et qui a été employée à toutes les opérations des Académiciens " envoyés au cercle polaire . . . . le degré 13 au dessus de 0; et c'est précisément " celui que le thermomètre de M. de *Reaumur* marquoit à Paris en 1735, lorsque notre Toise de fer fut " étalonnée sur celle du Châtelet par M. *Godin*.—*Mesure des Trois premiers Degrés du Méridien dans l'Hémi-* " *sphere Austral, par M. de la Condamine; à Paris*, 1751, *pp.* 75, 85.

" La Toise du Pérou et celle du nord ou du cercle polaire sont pareilles entre elles. Ce sont des règles plates " de fer poli, dont la largeur totale est de 17 à 18 lignes, et l'épaisseur de 4 lignes environ; leur longueur d'un " côté est de 2 pouces à peu près de plus de 6 pieds; elles sont coupées à chaque bout sur une largeur de 8 à 9 " lignes, et c'est la distance entre les vives arêtes de ces entailles qui a été prise pour la longueur de la toise: " les deux talons excédant d'environ un pouce à chaque extrémité, servent à garantir les arêtes des entailles de " tout choc. . . . . Ces deux toises ont été faites en 1735 par Langlois; celle du Pérou, " sous la direction de Godin; la seconde, sous la direction de Lacondamine, qui avoit alors le dessein de la laisser " en dépôt à l'Académie, pour avoir un modèle de celle qu'on emportoit au Pérou, et y avoir recours en cas qu'il " arrivât quelque accident à la première."—*Base du Système Métrique Décimal; tome troisième, p.* 405.

3.

The two direct copies of the *Toise of Peru* with which we are concerned were made by Fortin, of Paris; one in 1821 for M. Struve, the other in 1823 for M. Bessel. We shall here denote these bars by F_S and F_B. The authority of the former rests on the following certificate of the *Bureau des Longitudes*, signed by Arago :—

> " Je sousigné, membre de l'Institut et du bureau des longitudes,
> " certifie avoir comparé la Toise en fer construite par Fortin et
> " destinée à Monsieur Struve, à la Toise du Pérou, qui est con-
> " servée dans les archives de l'Observatoire Royal. Les deux
> " toises m'ont paru parfaitement égales; le comparateur dont je
> " me suis servi m'auroit fait connoitre une différence de la
> " deuxcentième partie d'un millimètre.
> " Paris, le 14 Novembre 1821. *ARAGO*."

This copy is preserved in the Observatory of Dorpat.

The authority of Bessel's Toise is stated in the following certificate :—

> " Le 31^me Août 1823, nous avons comparé, *Mr. ZAHRTMANN* et
> " moi, la toise en fer que *Mr. FORTIN* a construite pour
> " *Mr. BESSEL* de Königsberg, à l'étalon en fer de l'Observatoire
> " connu sous le nom de Toise du Pérou. Il nous a semblé
> " que la Règle de *Mr. BESSEL* est plus courte que l'étalon de
> " l'Observatoire de $\frac{1}{1278^{me}}$ de ligne (de un douze cent soixante-
> " dix-huitième de ligne).
>
> *F. ARAGO. ZAHRTMANN.*

This copy is preserved in the Observatory of Königsberg. It appears to be too short by $0^l\cdot00078$, or as it has generally been taken $0^l\cdot00080$.

The *Toise of Peru* at 16°·25 C., being $864^l\cdot0000$, we have, at the same temperature—

$$F_S = 864^l\cdot00000 \qquad (1)$$
$$F_B = 863\cdot99920$$

4.

The Standard to which all geodetical measuring apparatus in Russia are referred is a bar two toises in length, *à bouts*, designated **N**. Its length was ascertained by means of the toise F_S and an auxiliary toise **H**, whose length was also obtained from F_S. These comparisons are given at pp. LXXV, LXXVI, of the Introduction to M. Struve's account of the Russian " Arc of Meridian of 25° 20′ between the Danube and the Northern Ocean." Two of the toise bars compared at Southampton, viz., the Russian **P** and the Prussian T_{10} have been compared with **N**. Of the first of these Struve remarks :—" **P** a été confectionné à " l'atelier de l'Observatoire central en 1850, pour accompagner le nouvel appareil, destiné à " la mesure des deux bases les plus septentrionales. En effect **P** a servi, en 1850, pour la " mesure de la base d'Alten au Finmarken norvégien, et en 1851, pour la mesure de la " base d'Ofver-Tornea, en Laponie. En 1852, cet étalon a été envoyé en Bessarabie pour " être employé à la mesure de la base de Taschbunar, la plus méridionale de toutes. La " même année, 1852, cet étalon a été remis entre les mains du général Wrontschenko

" pour servir à la mesure des différentes bases des opérations géodésiques de la Russie méri-
" dionale, entreprises de la part du Dépôt topographique. Il se trouve à l'époque actuelle,
" 1859, entre les mains du colonel Wassiliew, dirigeant les opérations trigonométriques
" qui longent le Wolga," p. LXXVI. This bar **P**, then, which has been compared at Southampton, is one of great importance.

Of the Prussian toise, which M. Struve calls **B′**, as being a copy of Bessel's toise, he remarks :—" Le Gouvernement Prussien envoya à Poulkova, en 1852, une copie **B′** de la
" toise de l'Observatoire de Königsberg **B**, employée par Bessel dans ses expériences du
" pendule et dans sa mesure de degrés, toise qui depuis a servi d'unité pour toutes les
" opérations géodésiques de Prusse. Cette copie **B′**, en fer avec des boutons saillants en
" acier poli, est presque exactement égale à la dite toise de Bessel **B**, la différence étant
" **B** − **B′** = 0·00019, d'après l'inscription gravée sur **B′**. Ce **B′** a été directement
" comparé à **N**, à l'aide d'une toise auxiliare **H** . . . . La comparaison a été effectuée
" en déterminant alternativement **B′** − **H** et **N** − (**B′** + **H**)."

The following table contains the results of the comparisons of these bars by M. Struve, taken from page LXXIII of the work named above :—

Nom de l'étalon.	Longueur exprimée en lignes de la Toise de Fortin, celle-ci ayant la température + 13°·0 R.	Température de l'étalon déterminé, Réaum.	Erreur probable de la longueur, en unités de la cinquième décimale et relative à :	
			N	Fortin.
1. Étalon primitif à bouts . . . . **N**	1728·01249	13°·0	0	70
2. Copie à bouts . . . . **N′** = **P**	1727·99440	"	19	
9. Copie à bouts de la toise de Bessel . . . . . **B′**	863·99914	"	10	

5.

The Prussian and Belgian toises, Nos. 10 and 11, were compared in 1852 by General Baeyer with Bessel's toise. The comparisons will be found in the work entitled " *Compte
" rendu des Opérations de la Commission instituée par M. le Ministre de la Guerre, pour
" étalonner les règles qui ont été employées . . . . à la mesure des bases géodésiques
" belges.*" Bruxelles 1855. The results, page 49, are

$$\begin{aligned}\text{Toise No. 10} &= 863\overset{l}{\cdot}850674 + 0\overset{l}{\cdot}0091284\,t \\ \text{Toise No. 11} &= 863\cdot850213 + 0\cdot0091560\,t\end{aligned} \qquad (2)$$

where t is the temperature Centigrade. For $t = 16°\cdot25$, which is the same as 13° Reaumur or 61°·25 Fahrenheit,

$$\begin{aligned}\text{Toise No. 10} &= 863\overset{l}{\cdot}999011 \pm 0\overset{l}{\cdot}000109 \\ \text{Toise No. 11} &= 863\cdot998998 \pm 0\cdot000119\end{aligned} \qquad (3)$$

General Baeyer expresses his results thus, page 36 :—" Mesurée avec la toise de
" *Bessel*, la copie No. 10 est égale à 863^l·99901 avec l'erreur probable ± 0^l·00011. *Struve*
" a mesuré la longueur de cette copie à l'aide de la toise russe de *Fortin*, et il l'a trouvée
" égale à 863^l·99914, avec l'erreur probable ± 0^l·00010. D'où il suit que la toise de
" *Bessel*, mesurée avec la toise russe, égale 863^l·99933, tandis que le certificat de cette

" même toise lui assigne une longueur de $863^l \cdot 99920$. La différence est donc de $0^l \cdot 00013$,
" avec l'erreur probable $\pm\ 0^l \cdot 00015$.

"La toise No. 11 est précisément égale au No. 10. Sa longueur est donc, d'après la
" toise russe, de $863^l \cdot 99914$, avec l'erreur probable $\pm\ 0^l \cdot 00016$. Sa détermination d'après
" la toise de *Bessel* comporte l'erreur probable $\pm\ 0^l \cdot 00012$.

"La toise russe est donc de $0^l \cdot 00013$ plus longue que celle de Königsberg. Cette
" belle coincidence prouve que les comparaisons des diverses toises ont été faites avec une
" grande précision, et ne laissent rien à désirer au point de vue des opérations
" géodésiques."

The results in equation (2) were obtained on the assumption of Bessel's toise being represented by the equation

$$\mathbf{F}_D = 863^l \cdot 841161 + 0^l \cdot 0097255\, t \qquad (4)$$

where t is the temperature Centigrade. This rate of expansion was determined by *Bessel* himself between the years 1835 and 1837. An earlier investigation of the expansion of the same bar lead him to the result

$$\mathbf{F}_B = 863^l \cdot 835384 + 0^l \cdot 0100811\, t \qquad (5)$$

from either formula when $t = 16^\circ \cdot 25$, $\mathbf{F}_D = 863 \cdot 99920$.

6.

From the comparisons of the Prussian, Belgian, and Russian toises with the Ordnance toise ($\mathbf{T}_{10}$ $\mathbf{T}_{11}$ and $\mathbf{P}$ with $\mathbf{T}_0$) given in sections XI, XII, XXI, of the present work, we have

$\mathbf{T}_{10} = \mathbf{T}_0 - 154 \cdot 32 \pm \cdot 15$ . . . . page 134

$\mathbf{T}_{11} = \mathbf{T}_0 - 156 \cdot 10 \pm \cdot 27$ . . . . „ 150

$\mathbf{P} = 2\,\mathbf{T}_0 - 320 \cdot 48 \pm \cdot 31$ . . . . „ 266

all the bars being at the temperature of 62° Fahrenheit. It will be necessary to reduce these equations to what they would be if all the bars were at $61^\circ \cdot 25$ instead of $62^\circ \cdot 00$. There is no difficulty in this, and indeed the differences of length at the lower temperature have a slightly smaller probable error in each case, though the difference is scarcely sensible. For the temperature $61^\circ \cdot 25$ the above equations become

$$\begin{aligned} \mathbf{T}_{10} &= \mathbf{T}_0 - 154 \cdot 52 \pm 0 \cdot 15 \\ \mathbf{T}_{11} &= \mathbf{T}_0 - 156 \cdot 33 \pm 0 \cdot 27 \\ \mathbf{P} &= 2\,\mathbf{T}_0 - 321 \cdot 52 \pm 0 \cdot 31 \end{aligned} \qquad (6)$$

Such are the differences of length of the Prussian, Belgian, and Russian bars as compared with the Ordnance toise at Southampton, at the temperature of

13° *Reaumur* $= 16^\circ \cdot 25$ *Centigrade* $= 61^\circ \cdot 25$ *Fahrenheit.*

7.

The continental comparisons which we have referred to above, give the following equations—in which we use the letter $\mathfrak{T}$ to signify the length of *the toise*, that is the *Toise of Peru* at 13° R.

$$\begin{aligned}
F_S &= \mathfrak{T} \quad ' & (7)\\
F_D &= \mathfrak{T} - 0\cdot00080 \quad '\\
N &= 2\,F_S + 0\cdot01249 \pm 0\cdot00070\\
P &= N - 0\cdot01809 \pm 0\cdot00019\\
2\,T_{10} &= N - 0\cdot01421 \pm 0\cdot00020\\
T_{10} &= F_D - 0\cdot00019 \pm 0\cdot00011\\
T_{11} &= F_D - 0\cdot00020 \pm 0\cdot00012
\end{aligned}$$

Now $1'\cdot00 = \frac{1}{864}\,\mathfrak{T} = 2467\cdot03$ millionths of a yard. Making this substitution in equations (7) and adding to them equations (6) we have the following system:

$$\begin{aligned}
F_S &= \mathfrak{T} & (8)\\
F_D &= \mathfrak{T} - 1\cdot97\\
N &= 2\,F_S + 30\cdot81 \pm 1\cdot73\\
P &= N - 44\cdot63 \pm 0\cdot47\\
2\,T_{10} &= N - 35\cdot06 \pm 0\cdot49\\
T_{10} &= F_D - 0\cdot47 \pm 0\cdot27\\
T_{11} &= F_D - 0\cdot49 \pm 0\cdot30\\
T_{10} &= T_0 - 154\cdot52 \pm 0\cdot15\\
T_{11} &= T_0 - 156\cdot33 \pm 0\cdot27\\
P &= 2\,T_0 - 321\cdot52 \pm 0\cdot31
\end{aligned}$$

These equations trace the connection between the Ordnance toise and the *Toise of Peru* through the intervention of six other bars. Now let

$$\begin{aligned}
F_S &= \mathfrak{T} + x_1 & (9)\\
F_D &= \mathfrak{T} + x_2\\
N &= 2\,\mathfrak{T} + x_3\\
P &= 2\,\mathfrak{T} + x_4\\
T_{10} &= \mathfrak{T} + x_5\\
T_{11} &= \mathfrak{T} + x_6\\
T_0 &= \mathfrak{T} + x_7
\end{aligned}$$

Substituting these in (8) they become

$$
\begin{aligned}
x_1 \qquad\qquad\qquad &= 0 \\
x_2 + \quad 1 \cdot 97 &= 0 \\
x_3 - 2\,x_1 - \quad 30 \cdot 81 &= 0 \\
x_4 - \quad x_3 + \quad 44 \cdot 63 &= 0 \\
2\,x_5 - \quad x_3 + \quad 35 \cdot 06 &= 0 \\
x_5 - \quad x_2 + \quad 0 \cdot 47 &= 0 \\
x_6 - \quad x_2 + \quad 0 \cdot 49 &= 0 \\
x_5 - \quad x_7 + 154 \cdot 52 &= 0 \\
x_6 - \quad x_7 + 156 \cdot 33 &= 0 \\
x_4 - 2\,x_7 + 321 \cdot 52 &= 0
\end{aligned}
\qquad (10)
$$

We now solve these equations by the method of least squares; but without attempting to assign weights, we shall ascertain the values of $x_1\ x_2\ x_3\ x_4\ x_5\ x_6\ x_7$ which give the sum of the squares of the errors a minimum. The resulting equations are—

$$
\begin{aligned}
+5x_1 \quad - 2x_3 \qquad\qquad\qquad\qquad + \ 61 \cdot 62 &= 0 \\
+3x_2 \qquad\qquad - x_5 - x_6 \qquad + \quad 1 \cdot 01 &= 0 \\
-2x_1 \quad + 3x_3 - x_4 - 2x_5 \qquad\qquad - 110 \cdot 50 &= 0 \\
- x_3 + 2x_4 \qquad\qquad - 2x_7 + 366 \cdot 15 &= 0 \\
- x_2 - 2x_3 \qquad + 6x_5 \qquad - x_7 + 225 \cdot 11 &= 0 \\
- x_2 \qquad\qquad\qquad + 2x_6 - x_7 + 156 \cdot 82 &= 0 \\
- 2x_4 - x_5 - x_6 + 6x_7 - 953 \cdot 89 &= 0
\end{aligned}
\qquad (11)
$$

Putting $A_1\ A_2\ A_3 \ldots A_7$ for the absolute terms of these equations, we get—

$$
\begin{aligned}
296x_1 + 200\,A_1 + \ 96\,A_2 + 352\,A_3 + \ 336\,A_4 + 160\,A_5 + 128\,A_6 + 160\,A_7 &= 0 \\
296x_2 + \ 96\,A_1 + 200\,A_2 + 240\,A_3 + \ 256\,A_4 + 136\,A_5 + 168\,A_6 + 136\,A_7 &= 0 \\
296x_3 + 352\,A_1 + 240\,A_2 + 880\,A_3 + \ 840\,A_4 + 400\,A_5 + 320\,A_6 + 400\,A_7 &= 0 \\
296x_4 + 336\,A_1 + 256\,A_2 + 840\,A_3 + 1044\,A_4 + 402\,A_5 + 366\,A_6 + 476\,A_7 &= 0 \\
296x_5 + 160\,A_1 + 136\,A_2 + 400\,A_3 + \ 402\,A_4 + 239\,A_5 + 169\,A_6 + 202\,A_7 &= 0 \\
296x_6 + 128\,A_1 + 168\,A_2 + 320\,A_3 + \ 366\,A_4 + 169\,A_5 + 335\,A_6 + 206\,A_7 &= 0 \\
296x_7 + 160\,A_1 + 136\,A_2 + 400\,A_3 + \ 476\,A_4 + 202\,A_5 + 206\,A_6 + 276\,A_7 &= 0
\end{aligned}
\qquad (12)
$$

from which the actual values of $x_1 \ldots\ldots x_7$ are found to be—

$$
\begin{aligned}
x_1 &= - \quad 0 \cdot 07 & \qquad x_5 &= - \quad 2 \cdot 05 \\
x_2 &= - \quad 1 \cdot 90 & \qquad x_6 &= - \quad 2 \cdot 65 \\
x_3 &= + \ 30 \cdot 65 & \qquad x_7 &= + 153 \cdot 42 \\
x_4 &= - \ 14 \cdot 33 & &
\end{aligned}
\qquad (13)
$$

8.

We have now by the solution of ten equations containing seven unknown quantities,—these equations expressing the results of ten series of comparisons between seven bars of one or two toises in length amongst themselves and the *Toise of Peru*,—obtained the difference in length between that Standard and the Ordnance Toise at 61°·25 Fahrenheit. By equations (9) and (13) we have—

$$\mathbf{T}_0 = \mathfrak{T} + 153 \cdot 42 \qquad (14)$$

In order to estimate the probable error of this result, we must substitute the values just obtained of $x_1 \ldots x_7$ in equations (10), and we get the following system of errors :—

Comparisons.	Apparent Errors expressed in parts of the Yard and Toise.	
	$\frac{\mathfrak{Y}}{1000000}$	$\frac{\mathfrak{T}}{864}$
Struve's Toise with Toise of Peru ; by Arago	−0·07	−0·00003
Bessel's Toise with Toise of Peru ; by Arago and Zahrtmann	+0·07	+0·00003
Russian Normal Bar with Struve's Toise ; by Struve...	−0·03	−0·00001
Russian Normal Bar with its first copy **P** ; by Struve	−0·35	−0·00014
Prussian Toise, No. 10, with Russian Normal Bar ; by Struve	+0·31	+0·00013
Prussian Toise, No. 10, with Bessel's Toise ; by General Baeyer ...	+0·32	+0·00013
Belgian Toise, No. 11, with Bessel's Toise ; by General Baeyer	−0·26	−0·00010
Prussian Toise with Ordnance Toise, at Southampton	−0·95	−0·00039
Belgian Toise with Ordnance Toise, at Southampton	+0·26	+0·00010
Russian Bar **P** with Ordnance Toise, at Southampton	+0·35	+0·00014

It appears from this Table that the discrepancies among the different series of comparisons specified therein are remarkably small; the largest not amounting to the millionth part of a yard. Five of them are greater, and five less, than ± 0·30.

From the last of the equations (12) we gather that the weight of the determination of x_7 is somewhat greater than unity ; that is to say, our resulting value of $\mathbf{T}_0$ has at least as much weight as if it had been directly compared with the Toise of Peru. This is on the supposition of all the ten series of comparison being equally good : and we further get for the probable error of the value of $\mathbf{T}_0$, something less than ± 0·30. But the number of equations is not sufficiently large to admit of much precision, and we shall adopt ± 0·50 as a quantity which the probable error of our result does not exceed. Hence—

$$\text{At } 61^\circ\cdot25 \text{ F} \ldots\ldots\ldots \mathbf{T}_0 = \mathfrak{T} + 153\cdot42 \pm 0\cdot50 \qquad (15)$$

At page 144 we find that both bars being at 62°·00 Fahrenheit, $\mathbf{T}_0 = 2 \cdot 13167512\ \mathbf{Y}_{55}$; at 61°·25, we easily find from the data, pages 142, 143,

$$\mathbf{T}_0 = (2 \cdot 13167584 \pm 0\cdot00000021)\ \mathbf{Y}_{55} \qquad (16)$$

Taking the expansion of $\mathbf{Y}_{55}$ as 6·5145, (see pages 90 and 227) the difference of length of $\mathbf{Y}_{55}$ at 62°·00 and of the same bar at 61°·25 is 4·886 : putting, therefore, in the last equation $\mathbf{Y}_{55} - 4 \cdot 886$ in place of $\mathbf{Y}_{55}$ we have—

$$\mathbf{T}_0 = (2 \cdot 13166543 \pm 0\cdot00000021)\ \mathbf{Y}_{55} \qquad (17)$$

where $\mathbf{T}_0$ is at the temperature 61°·25 Fahr^t. and $\mathbf{Y}_{55}$ at the temperature 62°·00. This equation combined with (15) gives finally for the *Toise of Peru*,

$$\mathfrak{T} = (2{\cdot}13151201 \pm 0{\cdot}00000054)\ \mathbf{Y}_{55} \qquad (18)$$

9.

The corresponding values of the other toises resulting from the values of $x_1\ x_2 \ldots\ldots x_5$ equations (13) are as in the first column of the following Table :—

Toise.	Values resulting from the Solutions of Equations (8).	Value assigned by M. Arago.	Value assigned by M. Struve.	Value assigned by General Baeyer.	Temp^r of Bar.
					C
$\mathbf{F}_8$	863·99997	864·00000			16°·25
$\mathbf{F}_B$	863·99923	863·99920			"
N	1728·01242	... ...	1728·01249		"
P	1727·99419	... ...	1727·99440		"
$\mathbf{T}_{10}$	863·99917	... ...	863·99914	863·99901	"
$\mathbf{T}_{11}$	863·99893	... ...	... ...	863·99900	"

10.

There remains yet another determination of the value of the Toise in terms of the Yard,—that resulting from the comparison of the Indian Standard **B** (which we have called $\mathbf{I}_b$) with a double *Toise à traits.* From these comparisons which were made between the years 1847 and 1850 by M. Struve, at Poulkova, he arrived at the result that at 62°·00 Fahr^t. the length of the Indian Bar was

$$\mathbf{I}_b = 1351{\cdot}14398 \left(\frac{\mathbf{F}_8}{864}\right)$$

the toise of Fortin being at its normal temperature of 61°·25 Fahr^t. From this we get—

$$\mathbf{F}_8 = \frac{864}{1351{\cdot}14398}\ \mathbf{I}_b$$

We have shown that the length of $\mathbf{I}_b$ is 3·33331590 $\mathbf{Y}_{55}$, both bars being at 62°, hence

$$\mathbf{F}_8 = \mathfrak{T} = 2{\cdot}13151595\ \mathbf{Y}_{55}$$

This value of the toise is 3·94 millionths of a yard greater than that we have found from the comparisons of the Ordnance toise with the Russian, Prussian, and Belgian toises. The stated probable error of Struve's determination of the length of $\mathfrak{t}_b$ with reference to his normal bar **N** is $\pm$ 0^l·00044 or $\pm$ 1·09 millionths of a yard, which is about the same as our estimated probable error of $\mathfrak{t}_b$ with respect to $\mathbf{Y}_{55}$. The difference 3·94 is a small quantity when we consider the intricacy of the operations necessary to obtain the ratio of the toise and yard. We shall make no further use of this value of $\mathfrak{T}$; merely regarding it as an entirely independent corroboration of the result stated in equation (18).

11.

THE METRE.

The metre is by definition 443·296 "lignes" of the *Toise of Peru*.* Putting then $\mathfrak{M}$ for the length of the metre

$$\mathfrak{M} = \frac{443296}{864000}\,\mathfrak{T} \tag{19}$$

Substituting the value of $\mathfrak{T}$ from (18), we get the length of *The Metre*

$$\mathfrak{M} = (1.09362355 \pm 0.00000028)\,\mathbf{Y}_{55} \tag{20}$$

12.

From the observations recorded in Section IX. for the determination of the length of the Ordnance metre, it has been shown that its length is

$$1·09375344\ \mathbf{Y}_{55}$$

both bars being at 62° Fahrenheit. But if both bars be at 61°·25, we get

$$1·09375378\ \mathbf{Y}_{55}$$

* "Nous avons trouvé par cette méthode et par des calculs toujours faits par différens calculateurs, que la "comparaison de l'arc intercepté entre Dunkerque et Montjouy, et l'arc mesuré au Pérou, donne pour l'apla- "tissement de la terre $\frac{1}{334}$ . . . . D'après cette donnée nous avons calculé et toujours "par différentes méthodes le quart du méridien en employant l'arc intercepté entre Dunkerque et Montjouy, "et il en résulte que le quart du méridien est de 5130740 demi-modules ou de 2565370 modules, quantité dont "la dix millionième partie est 0·513074 demi-modules ou 0·256537 modules. Nous sommes donc d'avis, et voilà "en deux mots le résumé de tout notre travail, que, pour tirer de l'opération qui vient d'être faite en France et "en Espagne, le résultat le plus naturel et le plus vrai pour l'unité de mesure, il conviendra d'établir cette unité "nommée *mètre*, et qui est la dix millionième partie du quart du *méridien*, de 0·256537 module; ce qui puisque "le module est, comme nous l'avons dit au commencement de ce mémoire, la double toise, revient, selon les "anciennes mesures, à 3 pieds 11·296 lignes, en employant la toise du Pérou, à 13 degrés du thermomètre à "mercure divisé en 80 parties."

"La Commission, en suivant l'esprit du système métrique proposé par l'Académie et adopté par la loi, a choisi "la température de la glace fondante, ou ce que nous nommons le *zéro* de nos thermomètres; température con- "stante. C'est donc à cette température que l'étalon de platine a été tendu égal à 443^l·296 de la toise du "Pérou, cette toise étant supposée à 16¼°, comme il a été dit ci-dessus."—*Base du Système Métrique Décimal,* Tom. iii., *pp.* 432, 433, 642.

for the length of **OM**. Now the length of $\mathbf{Y}_{55}$ at $61^\circ \cdot 25$ is less than the length of the same bar at 62° by $4 \cdot 886$: putting therefore in the last equation, $\mathbf{Y}_{55} - 4 \cdot 886$ instead of $\mathbf{Y}_{55}$, we get

$$1 \cdot 09374844 \ \mathbf{Y}_{55} \qquad (21)$$

as the length of **OM** at $61^\circ \cdot 25$, while $\mathbf{Y}_{55}$ is at its normal temperature of $62^\circ \cdot 00$.

Also from equations (5), page 172, we find that both bars being at $61^\circ \cdot 25$, the *Royal Society's Platinum Metre* exceeds the Ordnance metre by $9 \cdot 98$, that is, its length is

$$1 \cdot 09375842 \ \mathbf{Y}_{55}$$

It would appear from the description given in the *Base du Système Métrique Décimal*, that the platinum bars which were to represent the *metre* at the temperature of melting ice, ($0^\circ \cdot 00$ C. $= 0^\circ \cdot 00$ R. $= 32^\circ \cdot 00$ F.), were laid off from the *Toise of Peru* at 13° Reaumur, allowance being made for the contraction of the bars, according to the rate of expansion of platinum as ascertained by *Borda*. At page 326, tom. iii., *Borda* states his results thus: that the expansion of platinum for one degree of the thermometer of Reaumur is $\frac{1}{92800}$. According to this the correction to the length of the platinum metres at $16^\circ \cdot 25$ C. $= 13^\circ$ R. would be—

$$\frac{13}{92800} \mathfrak{M}$$

which in parts of the yard $= 0 \cdot 00015320$.

Nothing is known as to the construction of the particular platinum metre we are considering; but it can only be assumed as most probable that it was constructed in the manner described, allowance being made for its expansion under 13° Reaumur. We must therefore deduct from the length of the Royal Society's metre at 13° R., $0 \cdot 00015320$, which leaves

$$1 \cdot 09360522 \ \mathbf{Y}_{55}$$

Now according to M. Arago's verification of this bar it is less than a metre by $17 \cdot 59$ thousandth parts of a millimetre, that is by $19 \cdot 24$ millionths of a yard. Hence, finally, adding this quantity, we get the value of *The Metre* as deduced from the *Royal Society's Platinum Metre*—

$$\mathfrak{M} = 1 \cdot 09362446 \ \mathbf{Y}_{55}$$

No probable error can be assigned, as there is no way of estimating the accuracy of M. Arago's measurement.

Comparing this with the value obtained through the toise, equation (20), we find the difference is only $0 \cdot 91$ millionths of a yard, an agreement equally remarkable and satisfactory.

13.

We shall now for convenience bring together all our results in the following table; taking the value of $\mathbf{Y}_{55}$ from equations (24) page 162.

Relative Lengths of Standards.

Standards.		Expressed in Terms of the Standard Yard. $\mathfrak{Y}$	Expressed in Inches. Inch $= \frac{1}{36}\mathfrak{Y}$	Expressed in Lines of the Toise. Line $= \frac{1}{864}\mathfrak{T}$	Expressed in Millimetres. Millimetre $= \frac{1}{1000}\mathfrak{M}$
The Yard		1·00000000	36·000000	405·34622	914·39180
Copy No. 55 of the Yard—at its Standard Temp^{e}.	62°·00F.	0·99999960	35·999986	405·34606	914·39143
Ordnance Standard Foot ,, ,, ,, ...	62·00	0·33333284	11·999982	135·11521	304·79681
Indian Standard Foot ,, ,, ,, ...	62·00	0·33333611	12·000100	135·11653	304·79980
Ordnance 10 ft. Bar O_1 ,, ,, ,, ...	62·00	3·33333717	120·000138	1351·15563	3047·97616
,, ,, O'_1 ,, ,, ,, ...	62·00	3·33335432	120·000755	1351·16259	3047·99184
Indian 10 ft. Bar I_s ,, ,, ,, ...	62·00	3·33340138	120·002450	1351·18166	3048·03488
,, ,, I_{II} ,, ,, ,, ...	62·00	3·33353284	120·007182	1351·23495	3048·15508
,, ,, I_b ,, ,, ,, ...	62·00	3·33331457	119·999324	1351·14647	3047·95550
Australian Standard O'_4 ,, ,, ,, ...	62·00	3·33330427	119·998954	1351·14230	3047·94608
,, ,, OI_6 ,, ,, ,, ...	62·00	3·33333747	120·000149	1351·15576	3047·97644
Ordnance Toise ,, ,, ,, ...	61·25	2·13166458	76·739925	864·06219	1949·17660
Ordnance Metre ,, ,, ,, ...	61·25	1·09374800	39·374928	443·34662	1000·11420
Royal Society's Metre à traits ,, ,, ...	32·00	1·09360478	39·369772	443·28857	999·98324
Prussian Toise No. 10 ,, ,, ,, ...	61·25	2·13150911	76·734328	863·99917	1949·03444
Belgian Toise No. 11 ,, ,, ,, ...	61·25	2·13150851	76·734306	863·99893	1949·03390
Russian Double Toise P ,, ,, ,, ...	61·25	4·26300798	153·468287	1727·99419	3898·05952
The Toise		2·13151116	76·734402	864·00000	1949·03632
The Metre		1·09362311	39·370432	443·29600	1000·00000

APPENDIX.

FIGURE OF THE EARTH.

The semiaxes of the spheroid resulting from the investigation of the Figure of the Earth in the *Account of the Principal Triangulation of Great Britain and Ireland* are expressed in feet of the Ordnance Survey ten-feet Standard Bar O_1. They are moreover dependent on a numerical ratio of this bar to the *Toise of Peru* deduced from Mr. Baily's comparison of the platinum metre (à traits) of the Royal Society with certain division lines on the Royal Astronomical Society's tubular scale, combined with comparisons by the late Lt. Murphy, R.E., between this last scale and the ten-feet Standard O_2, and another series of comparisons of O_2 with O_1. From these different comparisons the lengths of the metre and toise were concluded to be—

$$\text{Metre} = 3\cdot28087463 \text{ feet of } O_1$$

$$\text{Toise} = 6\cdot39454378 \text{ feet of } O_1$$

the second of these numbers being simply computed from the first by the legally defined ratio (443296 : 864000) of the metre and toise.

Although this determination of the length of the toise had a considerable *à priori* probable error as depending entirely upon observations on a bar so difficult to observe as the platinum metre, still it happens to be exceedingly near the truth, for the true length of the toise is now known with remarkable precision, the Belgian, Prussian, and Russian Standards giving almost the same result, as will be seen from the following table:—

Geodetic Standards at 13° R. = 16°·25 C. = 61°·25 F.	Length in Lines of the Toise of Peru at 13° R. = 16°·25 C. = 61°·25 F. according to		Length as determined at Southampton in feet of the Standard Yard.
	M. Struve.	General Baeyer.	
Russian double toise	1727·99440	....	12·78902289
Prussian toise..........	863·99914	863·99901	6·39453018
Belgian toise	....	863·99900	6·39452475

Whence the following entirely independent values of the toise in English feet:—

1st. Through the Russian Standard.................. 6·39453216 feet.
2nd. " Prussian " .... 6·39453703 "
3rd. " Belgian " .... 6·39453215 "

By combining the different comparisons made both in England and on the Continent on these bars, by the method of least squares, the final value of *THE TOISE* is—

$$6\cdot39453348 \text{ feet} : \log = 0\cdot8058088656$$

from which the greatest divergence of the three separate results above is only half a millionth of a toise, a difference corresponding to ten feet in the earth's radius.

The length of *THE METRE* deduced from the above by the fixed ratio of the metre and toise is—

$$3\cdot28086933 \text{ feet} : \log = 0\cdot5159889356$$

The length of the Ordnance Standard O_1 at 62°·00 F. in feet of the Standard Yard is—

10·00001151 feet.

The data of the problem of the Figure of the Earth are as follow, to commence with the French arc from Formentera to Dunkirk :—

Stations.	Astronomical Latitudes.			Distance of Parallels in Toises.	Distance of Parallels in Standard Feet.
	°	′	″		
Formentera	38	39	53·17	....	....
Montjouy	41	21	44·96	153673·61	982671·04
Barcelona	41	22	47·90	154616·74	988701·92
Carcassonne	43	12	54·30	259172·61	1657287·93
Pantheon	48	50	47·98	580312·41	3710827·13
Dunkirk	51	2	8·41	705257·21	4509790·84

The latitude of Formentera, as here given, is taken from the observations of M. Biot recorded and computed in the 3rd volume of his "*Traité Elémentaire d'Astronomie physique*," see page 509. With respect to the latitude of the Pantheon, the mean of 4532 observations gave 48° 50′ 48″·86 ("*Base du Système Métrique Décimal;*" ii., 413): this station is according to Delambre's geodetical connection, 560·99 toises or 35″·38 north of the south face of the Imperial Observatory in Paris. In the eighth volume of "*Annales de l'Observatoire Imperial de Paris*," page 317, we find several recent and closely agreeing determinations of the latitude of the south face of the Observatory, of which the mean is 48° 50′ 11″·71. Adding 35″·38 to this would give 48° 50′ 47″·09 for the latitude of the Pantheon; instead therefore of simply using the observed latitude of the Pantheon, we use the mean between this and the transferred latitude of the Observatory, viz., 48° 50′ 47″·98. The latitude of Dunkirk is that obtained from Ramsden's Zenith Sector; it agrees very closely with M. Leverrier's new determination (*Annales*, &c., viii., p. 256) 51° 2′ 8″·90,

The distance of the parallels of Dunkirk and Greenwich, deduced from the recent extension of the triangulation of England into France in 1862, is 161407·3 feet (of O_1) which is 3·9 feet greater than the distance deduced from Captain Kater's triangulation, and 3·2 feet less than the distance calculated by Delambre from General Roy's triangulation. This agreement of three entirely independent operations is highly satisfactory. The following table shows the data of the English arc with the distances of parallels in standard feet from Formentera :—

Stations.	Astronomical Latitudes.			Distance of Parallels in Feet of O_1 from Greenwich.	Distance of Parallels from Formentera in Standard Feet.
	°	′	″		
Formentera	..	..	..	....	....
Greenwich	51	28	38·30	....	4671198·3
Arbury	52	13	26·59	272639·0	4943837·6
Clifton	53	27	29·50	722864·3	5394063·4
Kellie Law	56	14	53·60	1742021·4	6413221·7
Stirling	57	27	49·12	2186122·5	6857323·3
Saxavord	60	49	37·21	3415618·5	8086820·7

The latitude assigned in this table to Saxavord is not the directly observed latitude, which is 60° 49′ 38″·58, for there are here a cluster of three points whose latitudes are astronomically determined, and if we transfer, by means of the geodetic connection, the latitude of *Gerth of Scaw* to Saxavord we get 60° 49′ 36″·59, and if we similarly transfer the latitude of *Balta* we get 60° 49′ 36″·46. The mean of these three is that entered in the above table.

For the Indian Arc, in Longitude 77° 40′, we have the following data:

Stations.	Astronomical Latitudes.	Distance of Parallels in feet of O_1.	Distance of Parallels in Standard Feet.
	° ′ ″		
Punnœ..........	8 9 31·132	....	....
Putchapolliam....	10 59 42·276	1029173·7	1029174·9
Dodagoontah.....	12 59 52·165	1756560·0	1756562·0
Namthabad......	15 5 53·562	2518373·4	2518376·3
Daumergida.....	18 3 15·292	3591784·3	3591788·4
Takalkhera......	21 5 51·532	4697324·1	4697329·5
Kalianpur.......	24 7 11·262	5794689·0	5794695·7
Kaliana.........	29 30 48·322	7755827·0	7755835·9

The data of the Russian Arc, (Long. 26° 40′,) taken from M. Struve's work are as below:

Stations.	Astronomical Latitudes.	Distance of Parallels in Toises.	Distance of Parallels in Standard Feet.
	° ′ ″		
Staro Nekrassowka....	45 20 2·94	....	....
Wodolui.............	47 1 24·98	96415·136	616529·81
Ssuprunkowzi........	48 45 3·04	194973·124	1246762·17
Kremenetz..........	50 5 49·95	271724·510	1737551·48
Belin...............	52 2 42·16	382943·521	2448745·17
Nemesch............	54 39 4·16	531753·042	3400312·63
Jacobstadt..........	56 30 4·97	637483·921	4076412·28
Dorpat.............	58 22 47·56	744764·484	4762421·43
Hogland............	60 5 9·84	842303·102	5386135·39
Kilpi-maki..........	62 38 5·25	988016·669	6317905·67
Tornea.............	65 49 44·57	1170810·973	7486789·97
Stuor-oivi..........	68 40 58·40	1334032·877	8530517·90
Fuglenœs..........	70 40 11·23	1447786·783	9257921·06

For the Arc measured by Sir Thomas Maclear, (Longitude 18° 30′,) we have—

Stations.	Astronomical Latitudes.	Distance of Parallels in Feet of O_1.	Distance of Parallels in Standard Feet.
	° ′ ″		
N. End.............	29 44 17·66	....	....
Heerenlogement Berg.	31 58 9·11	811506·8	811507·7
Royal Observatory....	33 56 3·20	1526385·1	1526386·8
Zwart Kop..........	34 13 32·13	1632581·4	1632583·3
Cape Point.........	34 21 6·26	1678373·8	1678375·7

And, finally, for the Peruvian Arc, in Longitude 281° 0′:

Stations.	Astronomical Latitudes.	Distance of Parallels in Toises.	Distance of Parallels in Standard Feet.
	° ′ ″		
Tarqui.............	−3 4 32·068	....	....
Cotchesqui.........	0 2 31·387	176875·5	1131036·3

There is no necessity here to explain the manner of applying the method of least squares to the Figure of the Earth. Let the polar semiaxis be

$$\frac{20855500}{1 + \dfrac{u}{10000}} \tag{1}$$

and supposing the earth to be an ellipsoid, let*

$$n = \frac{1}{590} + (v + w \cos 2\,\omega + z \sin 2\,\omega) \sin 10'' \tag{2}$$

where n is the ratio of the difference of the equatorial (in longitude ω east of Greenwich) and polar semiaxes to their sum. Let x_1 x_2 x_3 x_4 x_5 be the corrections to the observed latitudes of the southern points of the five arcs, then the corrections to the latitudes of the other points will be as follow:—

Formentera	...	...	...	...	...	x_1
Montjouy	− 0·822	+ 0·9711u	− 0·2217v	− 0·2217w	− 0·0039z	+ x_1
Barcelona	− 4·179	+ 0·9770u	− 0·2235v	− 0·2235w	− 0·0039z	+ x_1
Carcassonne	− 6·118	+ 1·6375u	− 0·4502v	− 0·4501w	− 0·0079z	+ x_1
Pantheon	− 7·831	+ 3·6647u	− 1·5292v	− 1·5290w	− 0·0267z	+ x_1
Dunkirk	− 6·554	+ 4·4529u	− 2·1045v	− 2·1042w	− 0·0367z	+ x_1
Greenwich	− 4·536	+ 4·6120u	− 2·2310v	− 2·2307w	− 0·0389z	+ x_1
Arbury	− 4·157	+ 4·8808u	− 2·4528v	− 2·4524w	− 0·0428z	+ x_1
Clifton	− 7·862	+ 5·3248u	− 2·8410v	− 2·8405w	− 0·0496z	+ x_1
Kellie Law	− 6·518	+ 6·3294u	− 3·8165v	− 3·8159w	− 0·0666z	+ x_1
Stirling	− 6·151	+ 6·7671u	− 4·2832v	− 4·2825w	− 0·0748z	+ x_1
Saxavord	− 3·991	+ 7·9782u	− 5·7009v	− 5·7000w	− 0·0995z	+ x_1
Punnœ	...	...	...	...	...	x_2
Putchapolliam	− 0·733	+ 1·0211u	+ 0·9078v	− 0·8250w	+ 0·3789z	+ x_2
Dodagoontah	+ 4·634	+ 1·7426u	+ 1·5167v	− 1·3783w	+ 0·6330z	+ x_2
Namthabad	− 1·217	+ 2·4980u	+ 2·1208v	− 1·9273w	+ 0·8851z	+ x_2
Daumergida	+ 0·544	+ 3·5622u	+ 2·9016v	− 2·6368w	+ 1·2109z	+ x_2
Takal Khera	+ 2·941	+ 4·6579u	+ 3·6076v	− 3·2784w	+ 1·5056z	+ x_2
Kalianpur	− 3·110	+ 5·7451u	+ 4·1966v	− 3·8137w	+ 1·7514z	+ x_2
Kaliana	+ 1·523	+ 7·6864u	+ 4·9324v	− 4·4823w	+ 2·0585z	+ x_2
Staro Nekrassowka	...	...	...	...	...	x_3
Wodolui	+ 4·008	+ 0·6086u	− 0·3283v	− 0·1960w	− 0·2633z	+ x_3
Ssuprunkowzi	+ 5·379	+ 1·2306u	− 0·7177v	− 0·4286w	− 0·5757z	+ x_3
Kremenetz	+ 0·498	+ 1·7148u	− 1·0588v	− 0·6323w	− 0·8493z	+ x_3
Belin	+ 2·770	+ 2·4162u	− 1·6100v	− 0·9614w	− 1·2914z	+ x_3
Nemesch	+ 2·376	+ 3·3544u	− 2·4526v	− 1·4646w	− 1·9673z	+ x_3
Jacobstadt	+ 4·810	+ 4·0206u	− 3·1224v	− 1·8646w	− 2·5045z	+ x_3
Dorpat	+ 1·060	+ 4·6966u	− 3·8615v	− 2·3059w	− 3·0974z	+ x_3
Hogland	+ 2·119	+ 5·3110u	− 4·5830v	− 2·7368w	− 3·6761z	+ x_3
Kilpi-maki	+ 1·297	+ 6·2286u	− 5·7464v	− 3·4315w	− 4·6093z	+ x_3
Tornea	+ 6·528	+ 7·3793u	− 7·3412v	− 4·3838w	− 5·8885z	+ x_3
Stour-oivi	+ 1·261	+ 8·4065u	− 8·8851v	− 5·3058w	− 7·1269z	+ x_3
Fuglenœs	+ 2·783	+ 9·1223u	− 10·0206v	− 5·9839w	− 8·0377z	+ x_3
North End	...	...	...	...	...	x_4
Heerenlogement Berg	+ 0·303	+ 0·8032u	+ 0·1672v	+ 0·1335w	+ 0·1006z	+ x_4
Royal Observatory	− 0·755	+ 1·5104u	+ 0·2471v	+ 0·1974w	+ 0·1487z	+ x_4
Zwart Kop	+ 0·832	+ 1·6156u	+ 0·2534v	+ 0·2024w	+ 0·1525z	+ x_4
Cape Point	− 0·322	+ 1·6608u	+ 0·2557v	+ 0·2042w	+ 0·1539z	+ x_4
Tarqui	...	...	...	...	...	x_5
Cotchesqui	+ 0·582	+ 1·1224u	+ 1·0852v	− 1·0061w	− 0·4065z	+ x_5

* *Memoirs of the Royal Astronomical Society*, vol. xxix, page 30.

The sum of the squares of these 40 corrections being made a minimum, we get the following equations for $u\ v\ w\ z$ after having eliminated $x_1\ x_2\ x_3\ x_4\ x_5$;

$$\begin{aligned} 0 &= -33{\cdot}0215 + 228{\cdot}0293u - 136{\cdot}2457v - 147{\cdot}5770w - 82{\cdot}7288z \\ 0 &= +13{\cdot}1070 - 136{\cdot}2457u + 188{\cdot}6212v + 95{\cdot}9509w + 114{\cdot}7473z \\ 0 &= +16{\cdot}9976 - 147{\cdot}5770u + 95{\cdot}9509v + 100{\cdot}0405w + 56{\cdot}4999z \\ 0 &= -4{\cdot}9223 - 82{\cdot}7288u + 114{\cdot}7473v + 56{\cdot}4999w + 88{\cdot}5808z \end{aligned} \quad (3)$$

writing A B C D for the absolute terms in these equations, they become on elimination,

$$\begin{aligned} 0 &= u + 0{\cdot}103665\,A - 0{\cdot}015079\,B + 0{\cdot}158926\,C + 0{\cdot}014982\,D \\ 0 &= v - 0{\cdot}015079\,A + 0{\cdot}033558\,B - 0{\cdot}034271\,C - 0{\cdot}035694\,D \\ 0 &= w + 0{\cdot}158926\,A - 0{\cdot}034271\,B + 0{\cdot}263234\,C + 0{\cdot}024921\,D \\ 0 &= z + 0{\cdot}014982\,A - 0{\cdot}035694\,B + 0{\cdot}024921\,C + 0{\cdot}055623\,D \end{aligned} \quad (4)$$

Substituting here the values of A B C D, we get—

$$\begin{aligned} u &= +0{\cdot}9932 \\ v &= -0{\cdot}5309 \\ w &= +1{\cdot}3455 \\ z &= +0{\cdot}8128 \end{aligned} \quad (5)$$

These values being substituted in equations (1), (2), give—

Polar semiaxis = 20853429 feet.

$$\text{Equatorial semiaxis} = 20923161 + 3189 \cos 2(\omega - 15^{\circ}{\cdot}34')$$

$$n = 0{\cdot}001669176 + {\cdot}00007621 \cos 2(\omega - 15^{\circ}{\cdot}34')$$

Elements of the Figure of the Earth.

Semiaxes.	Length in		
	Feet.	Toise.	Metres.
Major semiaxis = a, of equator (long. 15°·34′ E.)	20926350	3272537·3	6378294·0
Minor semiaxis = b, of equator (long. 105°·34′ E.)	20919972	3271540·1	6376350·4
Polar semiaxis = c, ...	20853429	3261133·8	6356068.1

$$\frac{a-c}{c} = \frac{1}{285{\cdot}97} : \frac{b-c}{c} = \frac{1}{313{\cdot}38} : \frac{a-b}{c} = \frac{1}{3269{\cdot}5}$$

The length of the meridian quadrant passing through Paris, is . . . . . . 10001472·5 metres,

and the minimum quadrant, in longitude 105° 34′ is . . . . 10000024·5 metres

The corrections to the latitudes of the 40 Astronomical Stations computed from the above values of u v w z are as in the following Table :—

Stations.	Corrections.	Stations.	Corrections.
	″		″
Formentera	+2·7389	Staro Nekrassowka .	−3·3105
Montjouy	+2·6977	Wodolui	+0·9985
Barcelona	−0·6549	Ssuprunkowzi	+2·6272
Carcassonne	−2·1257	Kremenetz	−2·0882
Pantheon	−2·7193	Belin	+0·3709
Dunkirk	−1·1360	Nemesch	+0·1297
Greenwich	+0·9353	Jacobstadt	+2·6062
Arbury	+1·3976	Dorpat	−1·1557
Clifton	−2·1881	Hogland	−0·1535
Kellie Law	−0·6546	Kilpi-maki	−1·1397
Stirling	−0·2395	Tornea	+3·7599
Saxavord	+1·9489	Stour-oivi	−1·9142
		Fuglenœs	−0·7310
Punnœ	−0·3454		
Putchapolliam	−1·3484	North End	−1·3127
Dodagoontah	+3·8740	Heerenlogement Berg	−0·0393
Namthabad	−2·0812	Royal Observatory ..	−0·3123
Daumergida	−0·3676	Zwart Kop	+1·3856
Takal Khera	+2·1190	Cape Point	+0·2789
Kalianpur	−3·6854	Tarqui	+0·2818
Kaliana	+1·8351	Cotchesqui	−0·2818

The sum of the squares of these corrections is 138·3020: hence the probable error of a single latitude determination is

$$\pm\ 0{\cdot}674\sqrt{\frac{138{\cdot}3020}{40-9}} = \pm\ 1{\cdot}423$$

Spheroid of Revolution.

If in the first two of the equations (3) we made $w = 0$, $z = 0$, the values of u and v resulting from those two equations would determine the *Spheroid of Revolution* best representing the geodetic measurements. In this case we get

$$0 = u + 0{\cdot}0077151\,A + 0{\cdot}0055773\,B$$

$$0 = v + 0{\cdot}0055773\,A + 0{\cdot}0093270\,B$$

or,

$$u = +\ 0{\cdot}18172$$

$$v = +\ 0{\cdot}06177$$

And from these values there result

Semiaxes.	Length in		
	Feet.	Toises.	Metres.
Equatorial semiaxis $= a$	20926062	3272492·3	6378206·4
Polar semiaxis $= b$	20855121	3261398·4	6356583·8
$\frac{b}{a} = \frac{293{\cdot}98}{294{\cdot}98}$			

In this figure, however, the sum of the squares of the latitude corrections is 153·9939 against 138·3020 in the figure of three unequal axes.

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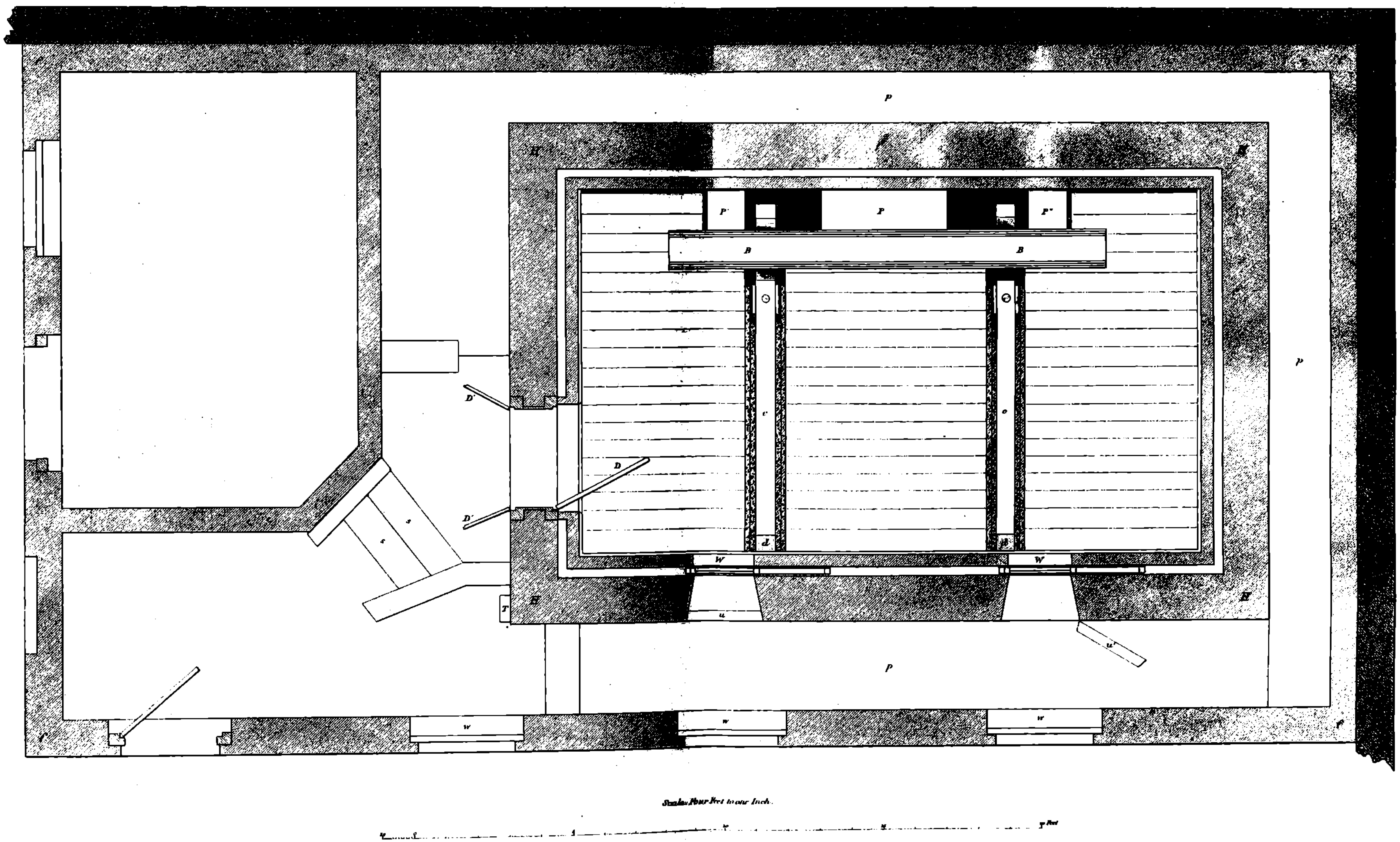
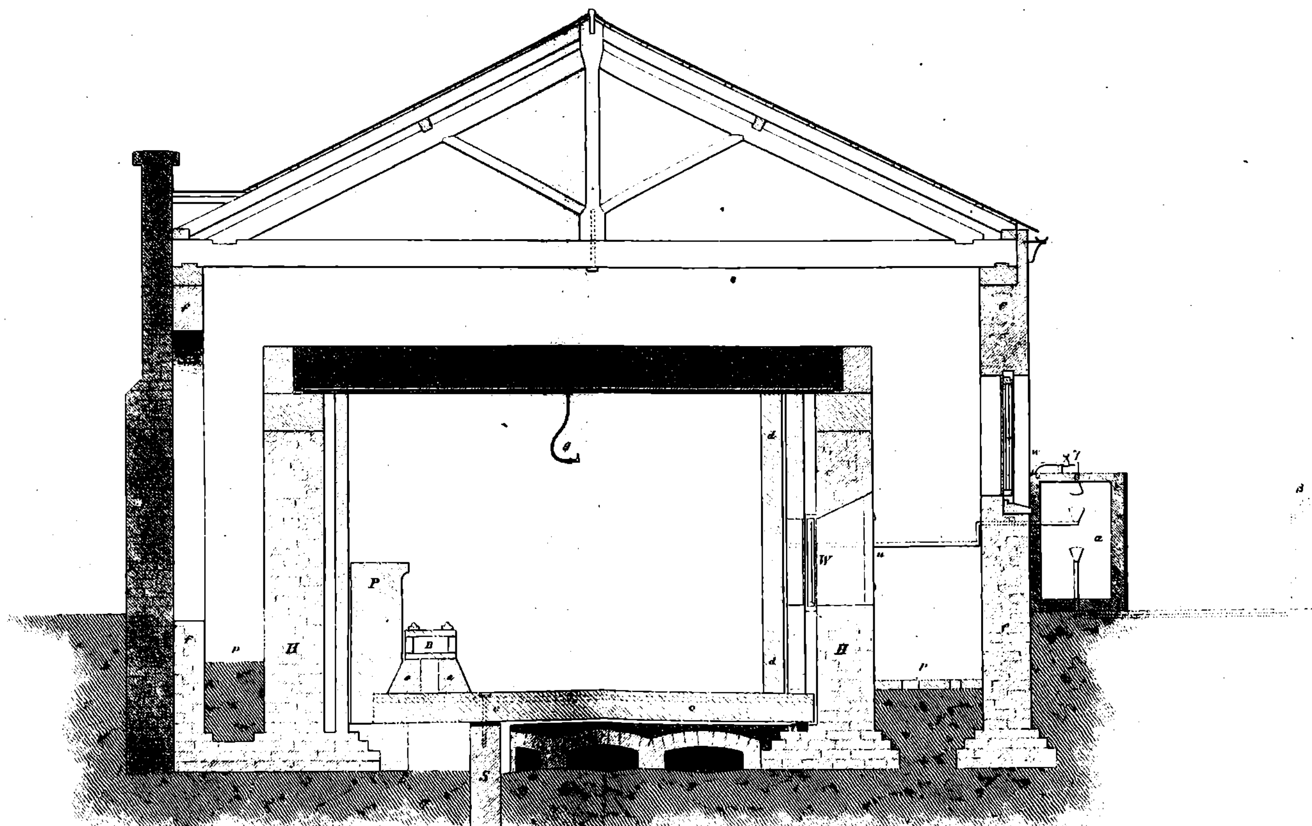
PLAN OF THE BAR ROOM.



Scale, Four Feet to one Inch.

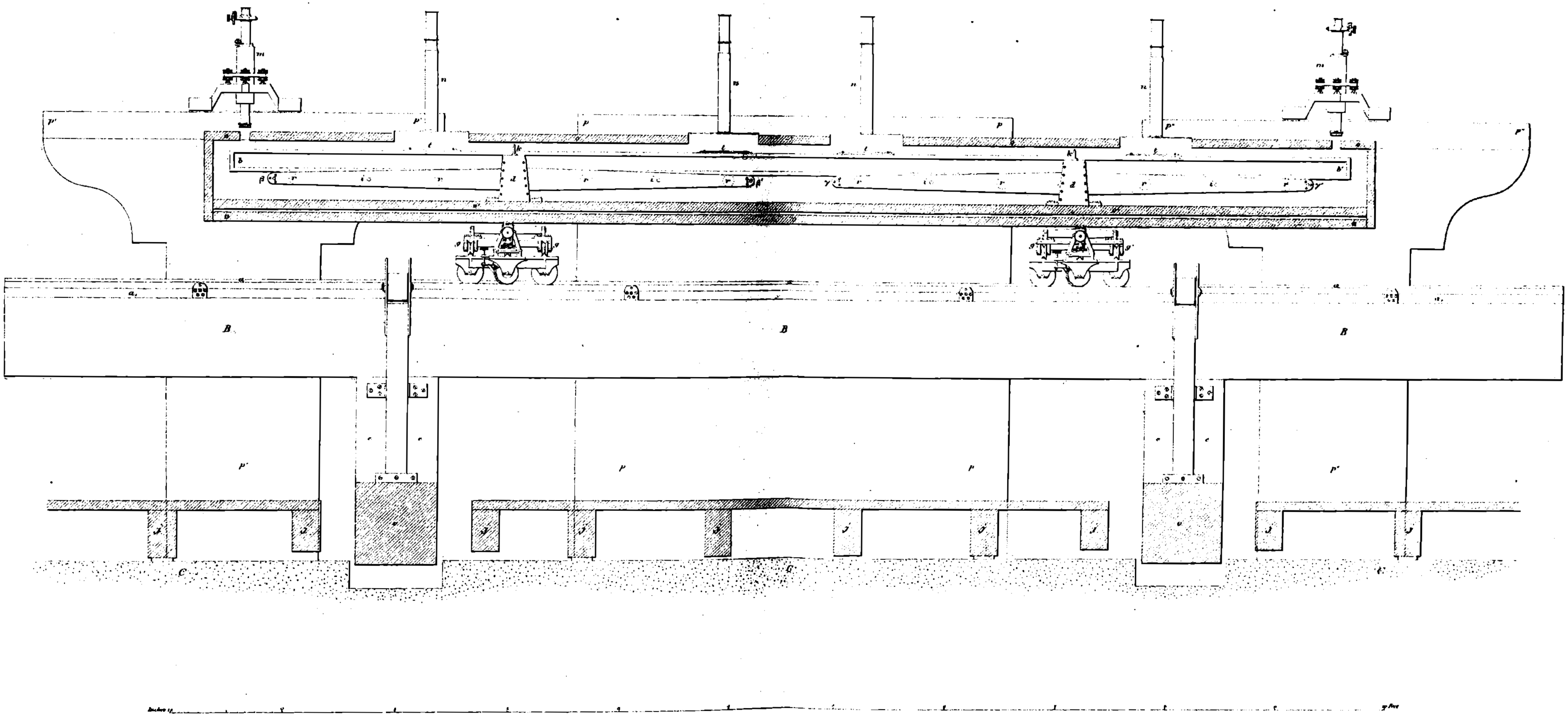
TRANSVERSE SECTIONAL ELEVATION OF THE BAR ROOM.



FRONT VIEW OF THE COMPARING APPARATUS.

with a Ten Feet Bar under Comparison

(The Floor of Room shewn in Section and Side of the Box supposed Removed)



END ELEVATION OF COMPARING APPARATUS.

End of Box supposed Removed

Scale ½

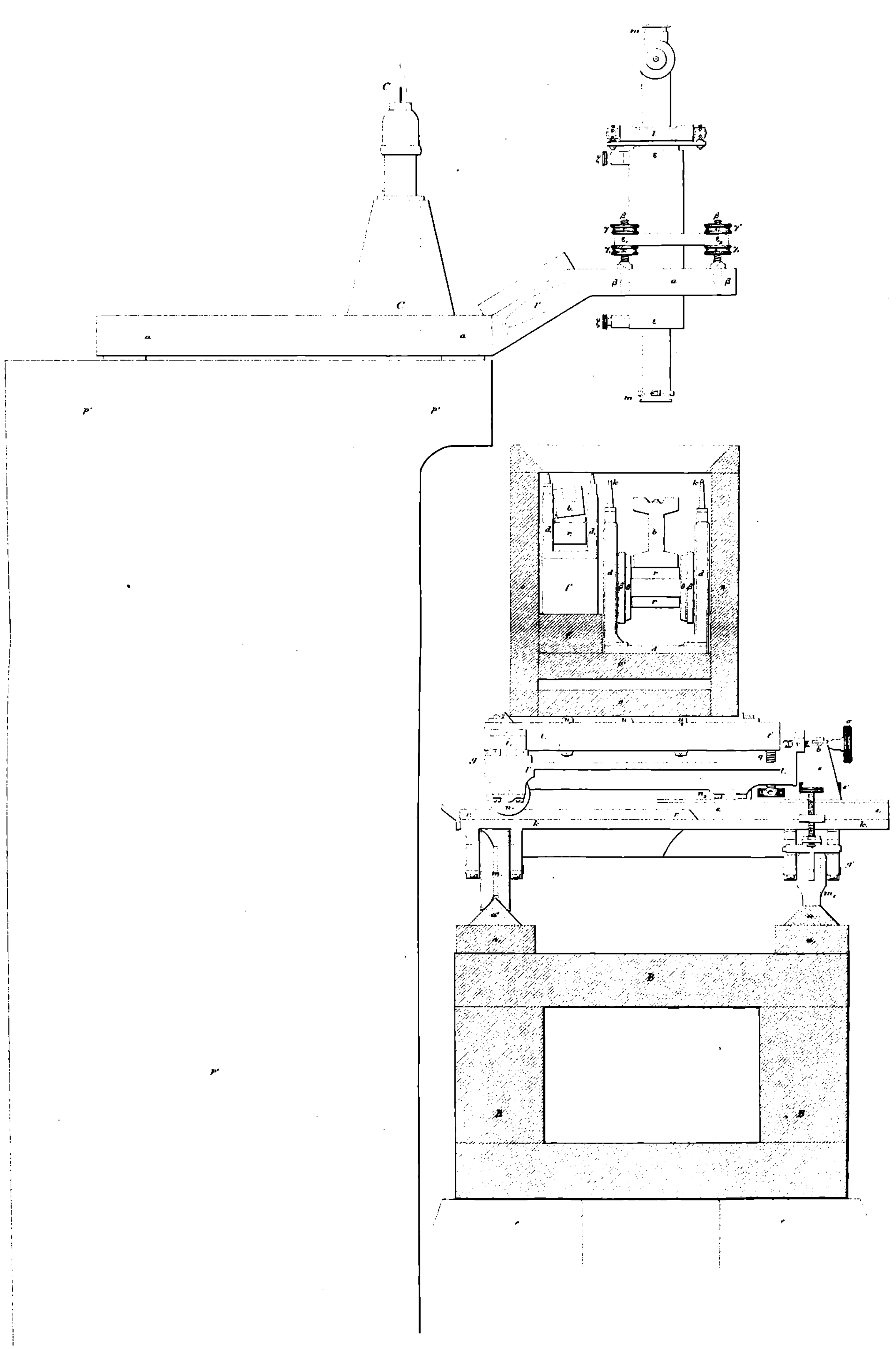


Fig. 1

Plan of Carriage.

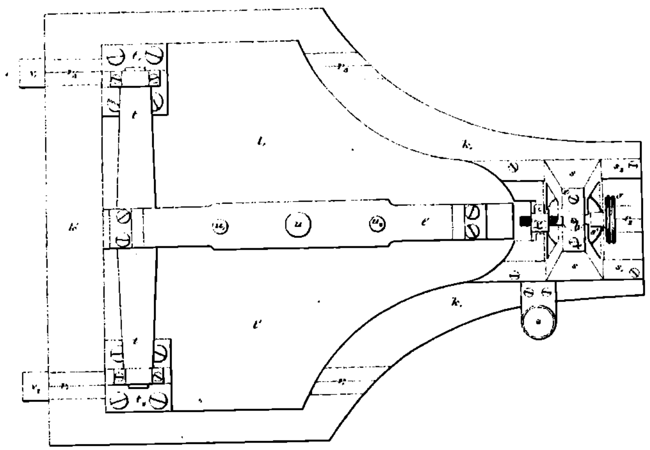


Fig. 2

Rear Elevation of Carriage

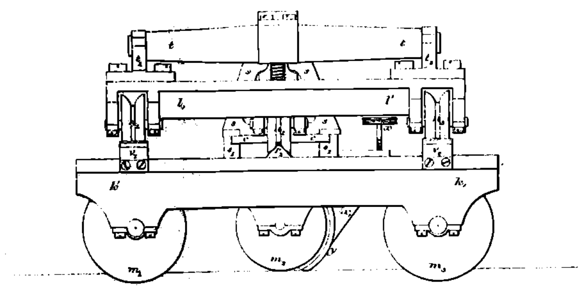
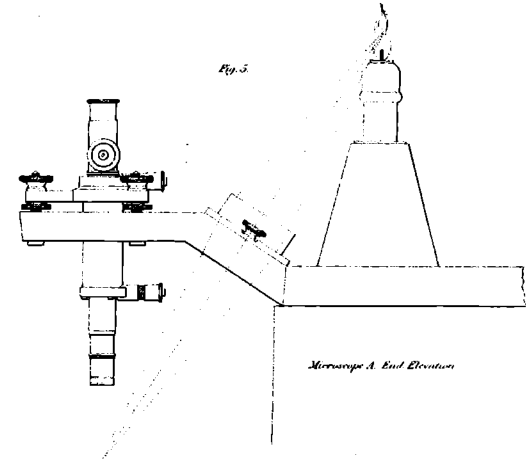
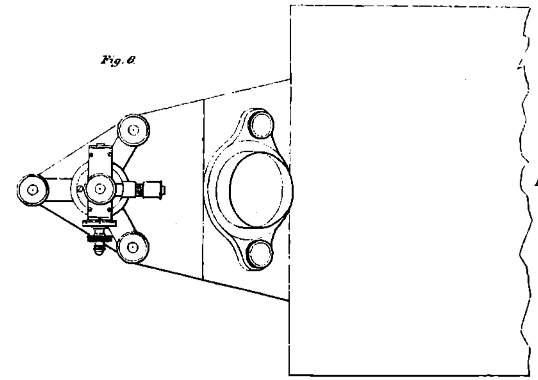


Fig. 5.



Microscope A. End Elevation

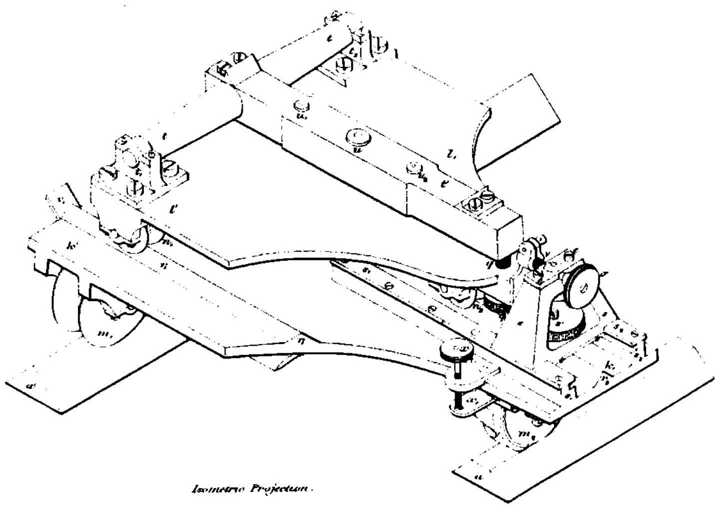
Fig. 6



Plan

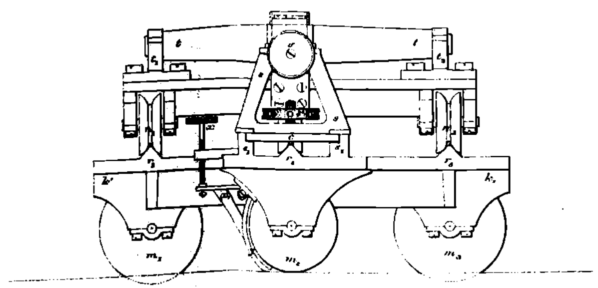
Scale of Figs. 1.2.3.4 to ¼

Fig. 3.



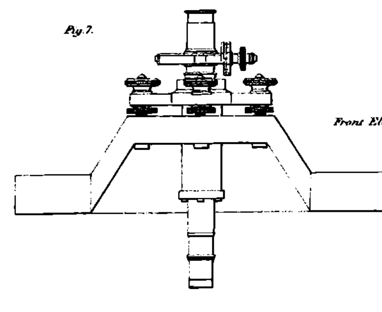
Isometric Projection.

Fig. 4.



Front Elevation

Fig. 7.



Front Elevation.

Scale of Figs 5.6.7 to ½

Isometric Projection of Microscope with Cast Iron Stand.

Fig. 1.

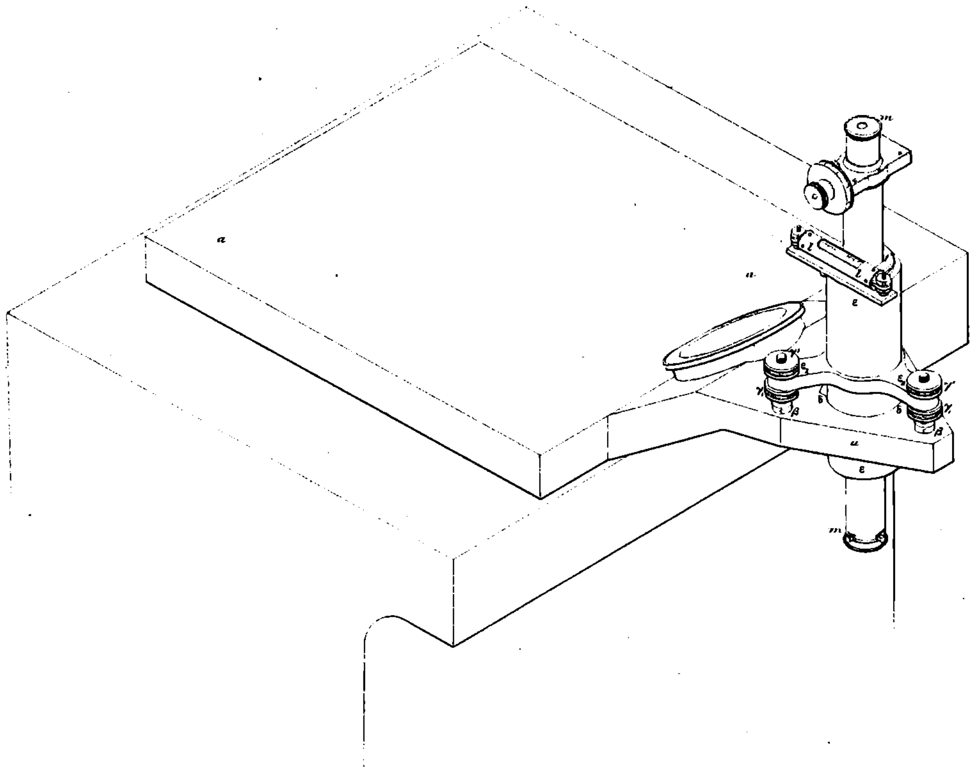


Fig. 2.

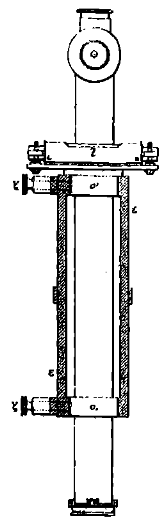


Fig. 3.

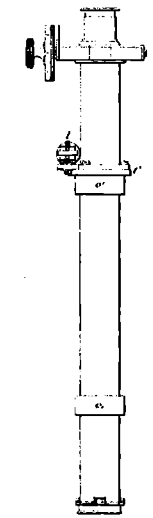
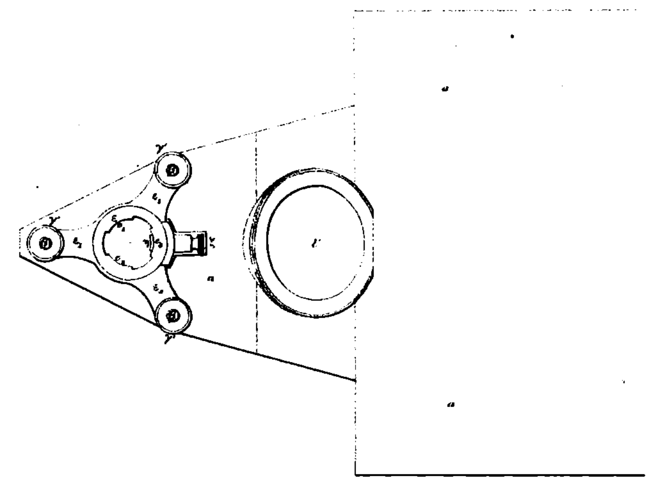
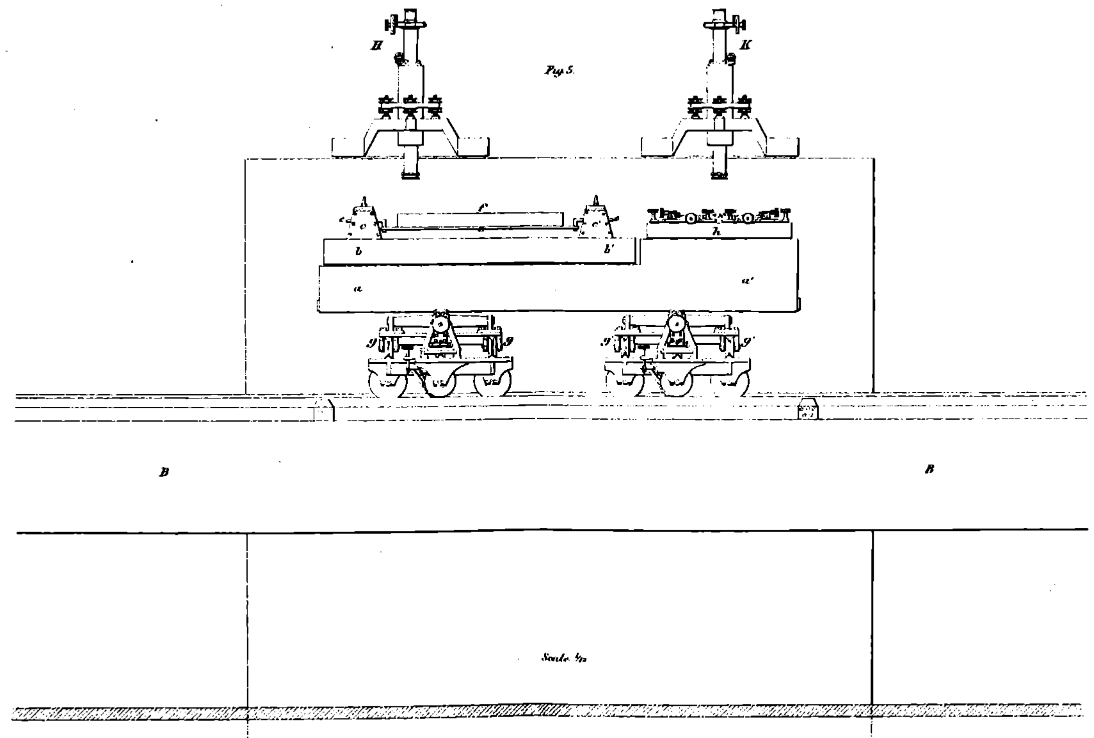


Fig. 4.



Scale of Figs. 1. 2. 3. 4. is ½

Fig. 5.



Scale 1/12

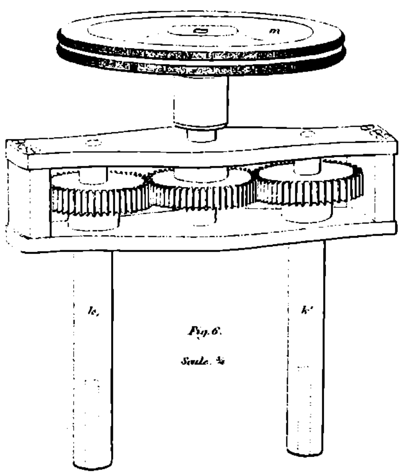
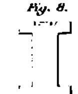


Fig. 6.

Scale. ½

Fig. 8.



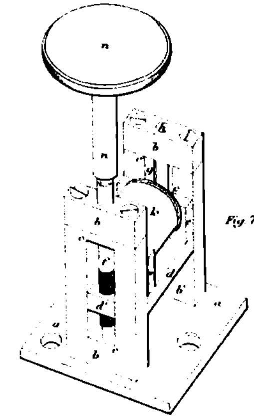


Fig. 7.

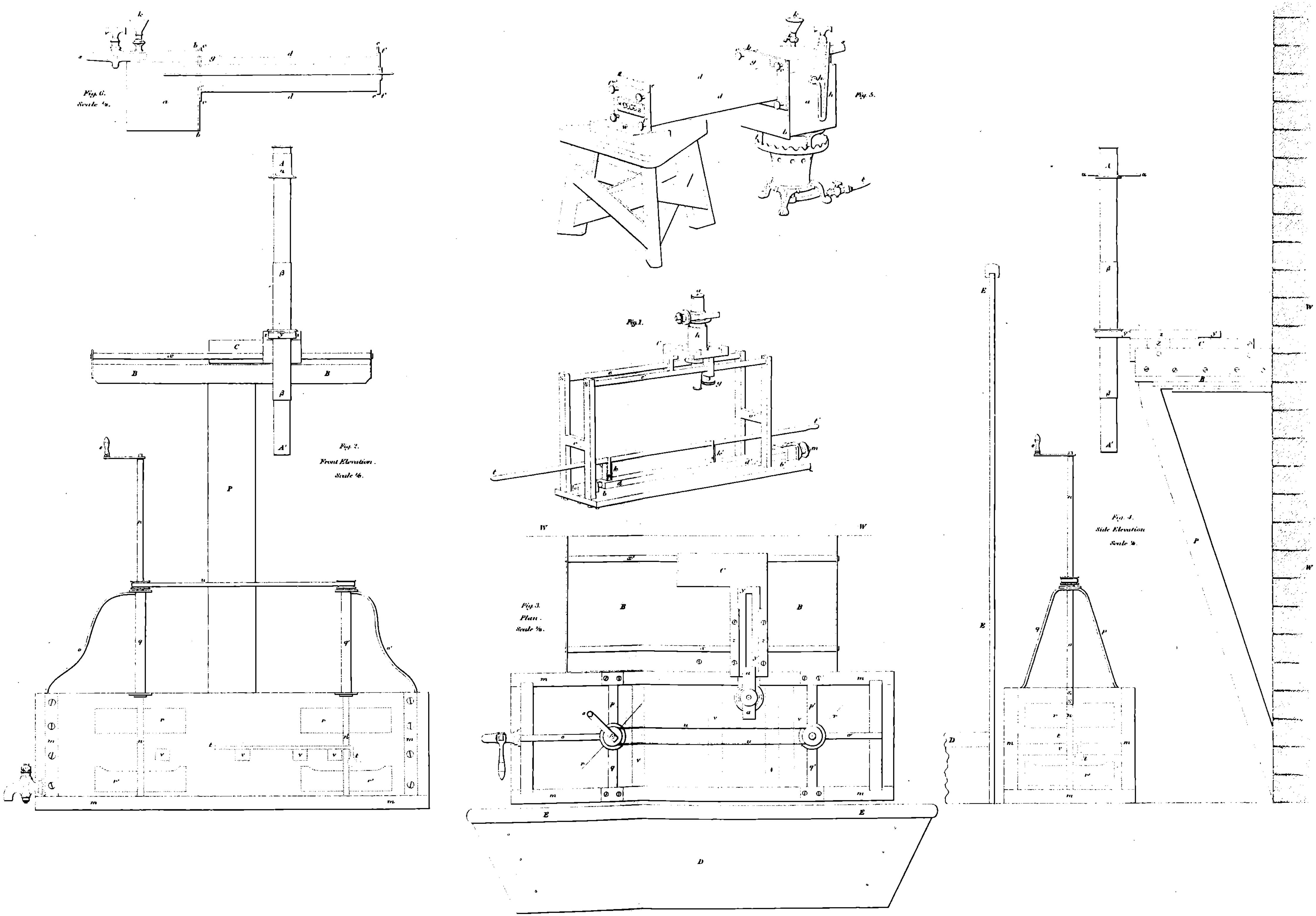


Fig. 6.
Scale 1/8.
Fig. 5.
Fig. 1.
Fig. 2.
Front Elevation.
Scale 1/6.
Fig. 3.
Plan.
Fig. 4.
Side Elevation.

Plate. VIII.

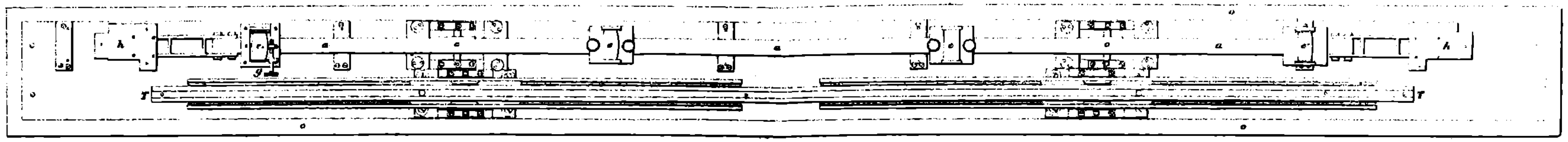


Fig. 1

Scale ½

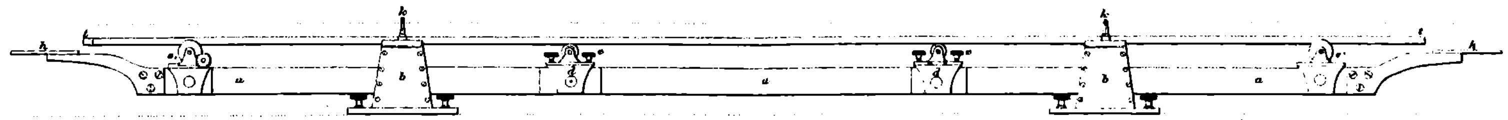
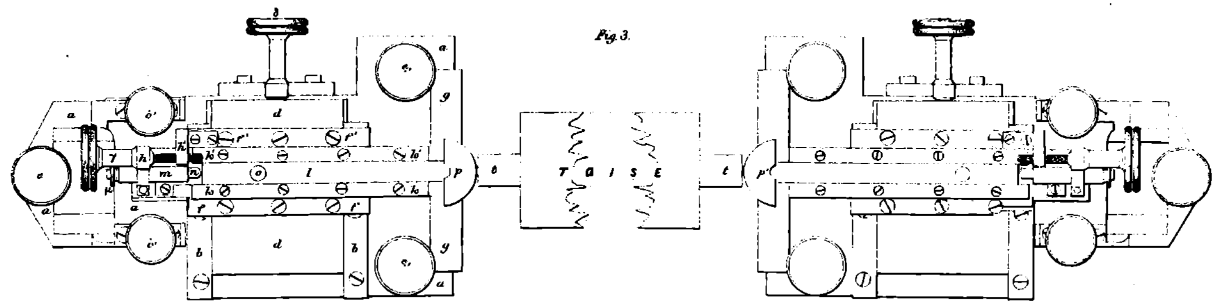


Fig. 2.

Fig. 3.



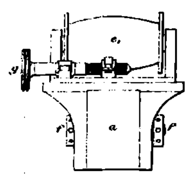


Fig. 4.

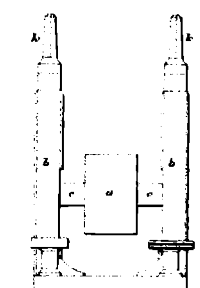
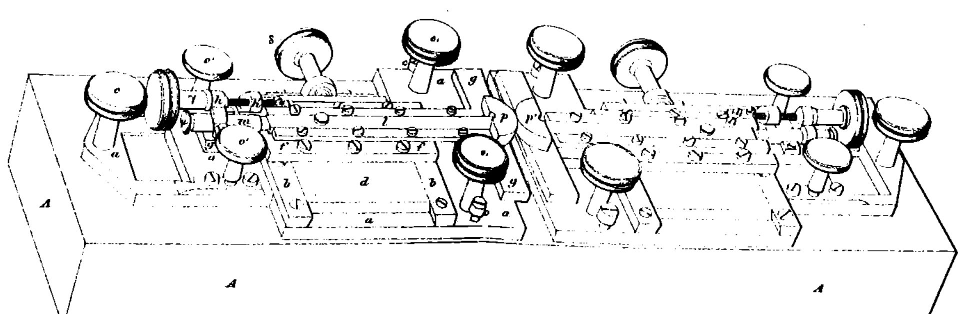


Fig. 5

Fig. 6.



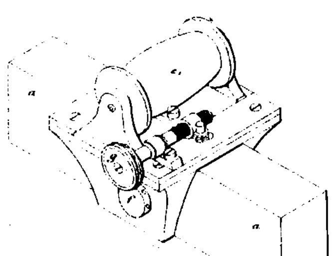


Fig. 7.

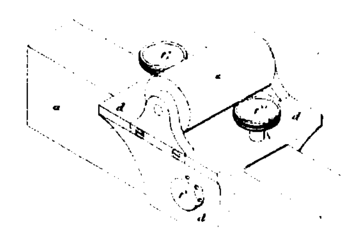


Fig. 8